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New Optimised Estimators for the Primordial Trispectrum

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ABSTRACT

Cosmic microwave background studies of non-Gaussianity involving higher-order multispectra can distinguish between early universe theories that predict nearly identical power spectra. However, the recovery of higher-order multispectra is difficult from realistic data due to their complex response to inhomogeneous noise and partial sky coverage, which are often difficult to model analytically. A traditional alternative is to use one-point cumulants of various orders, which collapse the information present in a multispectrum to one number. The disadvantage of such a radical compression of the data is a loss of information as to the source of the statistical behaviour. A recent study by Munshi & Heavens (2009) has shown how to define the skew spectrum (the power spectra of a certain cubic field, related to the bispectrum) in an optimal way and how to estimate it from realistic data. The skew spectrum retains some of the information from the full configuration-dependence of the bispectrum, and can contain all the information on non-Gaussianity. In the present study, we extend the results of the skew spectrum to the case of two degenerate power-spectra related to the trispectrum. We also explore the relationship of these power-spectra and cumulant correlators previously used to study non-Gaussianity in projected galaxy surveys or weak lensing surveys. We construct nearly optimal estimators for quick tests and generalise them to estimators which can handle realistic data with all their complexity in a completely optimal manner. Possible generalisations for arbitrary order are also discussed. We show how these higher-order statistics and the related power spectra are related to the Taylor expansion coefficients of the potential in inflation models, and demonstrate how the trispectrum can constrain both the quadratic and cubic terms.

Key words: : Cosmology– Cosmic microwave background– large-scale structure of Universe – Methods: analytical, statistical, numerical

1 INTRODUCTION

The inflationary paradigm which solves the flatness, horizon and monopole problem makes clear predictions about the generation and nature of density perturbations (Guth 1981; Starobinsky 1979; Linde 1982; Albrecht & Steinhardt 1982; Sato 1981). Inflationary models predict the statistical nature of these fluctuations, which are being tested against data from a range of recent cosmological observations, including the recently-launched all-sky cosmic microwave background (CMB) survey Planck[1]. Various ground-based and space-based observations have already confirmed the generic predictions of inflation, including a flat or nearly flat universe with nearly scale-invariant adiabatic perturbations at large angular scales. Several planned missions are also targeting the detection of the gravitational wave background through polarisation experiments - another generic prediction of inflationary models. The other major prediction of the inflationary scenarios is the nearly Gaussian nature of these perturbations. In the standard slow-roll paradigm, the scalar field responsible for inflation fluctuates with a minimal amount of self interaction which ensures that any non-Gaussianity generated during the inflation through self-interaction would be small (Salopek & Bond 1990, 1991; Falk et al. 1993; Gangui et al. 1994; Acquaviva et al. 2003; Maldacena 2003). See Bartolo, Matarrese & Riotto (2006) for a recent review and more detailed discussion. Any detection of non-Gaussianity would therefore be a measure of self-interaction or non-linearities involved, which can come from various alternative scenarios such as curvaton mechanism, warm inflation, ghost inflation as well as string theory inspired D-cceleration and Dirac Born Infield (DBI) models of inflation (Linde & Mukhanov 1997; Gupta, Berera & Heavens 2002; Lyth, Ungarelli & Wands 2003).

Early observational work on detection of primordial non-Gaussianity, from COBE (Komatsu et al. 2002) and MAXIMA (Santos et al. 2003) was followed by much more accurate analysis with WMAP[2](Komatsu et al. 2003; Creminelli et al. 2007; Spergel et al. 2007). Optimised 3-point estimators were introduced by Heavens (1998), and have been successively developed (Komatsu, Spergel & Wandelt 2005; Creminelli et al. 2006; Creminelli, Senatore, & Zaldarriaga 2007; Smith, Zahn & Dore 2007; Smith & Zaldarriaga 2006). Indeed, now an estimator for f_{NL} which saturates the Cramer-Rao bound has been found, capable of treating partial sky coverage and inhomogeneous noise (Smith, Senatore & Zaldarriaga 2009). The

[1] http://www.rssd.esa.int/index.php?project=Planck
[2] http://map.gsfc.nasa.gov/

recent claim of a detection of non-Gaussianity in WMAP data (Yadav & Wandelt 2008) has given a tremendous boost to the study of primordial non-Gaussianity, as it can lift the degeneracy between various early-universe theories which predict nearly the same primordial power spectrum. Most detection strategies focus on lowest order in non-Gaussianity, i.e. the bispectrum or three-point correlation functions (Komatsu, Spergel & Wandelt 2005; Creminelli 2003; Creminelli et al. 2006; Medeiros & Contaldo 2006; Cabella et al. 2006; Liguori et al. 2007; Smith, Senatore & Zaldarriaga 2009). This is primarily because of a decrease in signal-to-noise as we move up in the hierarchy of correlation functions - the higher-order correlation functions are more dominated by noise than their lower-order counterparts. Another related complication arises from the necessity to optimise such estimators, and the impact of inhomogeneous noise and partial sky coverage is always difficult to include in such estimates.

The recent study by Munshi & Heavens (2009) suggested the possibility of finding optimised cumulant correlators associated with higher-order multi-spectra in the context of CMB studies. These correlators are well-studied in the context of projected surveys such as projected galaxy surveys or in the context of weak lensing studies using simulated maps (Munshi 2000; Munshi & Jain 2000, 2001). Early studies involving cumulant correlators focused mainly on understanding gravity-induced clustering in collisionless media and were widely employed in many studies involving numerical simulations. Cumulant correlators are multipoint correlation functions, collapsed to two points Although they are two-point correlators, they carry information on the corresponding higher-order correlation functions. Due to their *reduced* dimensionality they do not carry all the information that is encoded in higher-order correlation function, but they carry more than their one-point counterparts, namely the moments of the probability distribution function which are often used as clustering statistics (Bernardeau et al. 2002).

One of the reasons to go beyond the lowest order in non-Gaussianity was pointed out by many authors, including. e.g., Riquelmel & Spergel (2006). At smaller angular scales, the secondary effect may dominate (Spergel & Goldberg 1999a,b), in direct contrast to larger angular scales where the anisotropies are generated mainly at the surface of the last scattering. These secondary perturbations are produced by interaction of CMB photons at much lower redshift with the intervening large-scale matter distribution. Such effects will be directly observable with Planck. The deflection of CMB photons by the large-scale mass distribution offers the possibility of studying the statistics of density perturbations in an unbiased way and provide clues to growth of structure formation for most part of the cosmic history. Weak lensing of the CMB can provide valuable information for constraining neutrino mass, dark energy equation of state and also has the potential to assist detection of primordial gravitation waves through CMB polarisation information; see e.g. Lewis & Challinor (2006) for a recent review. However weak lensing studies using the CMB need to address the contamination produced by other secondaries such as the thermal Sunyaev Zéldovich (tSZ) effects and kinetic Sunayev Zéldovich effect (kSZ) as well as by point sources, Although it is believed that these contaminations are not so important in case of polarisation studies. It was pointed out in Riquelmel & Spergel (2006) that a real space statistic such as $\langle \delta T^3(\hat{\Omega})\delta T(\hat{\Omega}')\rangle_c$ (which is a cumulant correlator of order four) can be used to separate the kSZ effect or Ostriker-Vishniac (OV) effect from the lensing effect as the lensing contribution cancels out at the lowest order. It was shown that, in addition to quantifying and controlling the kSZ contamination of lensing statistics such statistics could as well play a very important role in providing new insight into the history of reionization. In a completely different context it was shown that this estimator also has use for testing models of primordial non-gaussianity using redshifted 21cm observations (Cooray 2006; Cooray, Li & Melchiorri 2008). While cumulant correlators at third order $\langle \delta T^2(\hat{\Omega})\delta T(\hat{\Omega}')\rangle_c$ can provide information regarding the non-Gaussianity parameter $f_{\rm NL}$, fourth-order statistics such as $\langle \delta T^2(\hat{\Omega})\delta T^2(\hat{\Omega}')\rangle_c$ can go beyond the lowest order non-Gaussianity by putting constraints on the next-order parameter $g_{\rm NL}$ (to be introduced in later sections), albeit at lower signal-to-noise. However with ongoing CMB missions such as Planck the situation will improve and developing optimal methods for such higher-order cumulant correlators is the first step in this direction. There are now several studies which provide independent estimates of $f_{\rm NL}$ however we still lack such constraints for $g_{\rm NL}$. This clearly is related to the fact that in typical models $g_{\rm NL}$ is expected to be small, $g_{\rm NL} \le r/50$, where r is the scalar-to-tensor ratio (Serry, Lidsey & Sloth 2008). We also note that various studies have pointed out the link between the non-Gaussianity analysis and the estimators which test the anisotropy of the primordial universe. We plan to address these issues in a related publication. Several inflationary models provide direct consistency relations between $f_{\rm NL}$ and $g_{\rm NL}$, e.g. $g_{NL} = (6 f_{\rm NL}/5)^2$ (in some publications it is also denoted as f_2 or τ_{NL} e.g. (Hu & Okamoto 2002; Cooray 2006; Serry, Lidsey & Sloth 2008)). Testing of these consistency relations can give valuable clues to the mechanism behind the generation of initial perturbations. However, a difficulty for methods designed to detect non-Gaussianity in the CMB is that other processes can contribute, such as gravitational lensing, unsubtracted point sources, and imperfect subtraction of galactic foreground emission discussed by, e.g., Goldberg & Spergel (1999); Cooray (2000); Verde & Spergel (2002); Castro (2004); Babich & Pierpaoli (2008).

While the main motivation in this work is to study the primordial trispectrum, we note that mode-mode coupling resulting from weak lensing of the CMB produces a trispectrum which has been studied using power-spectra associated with $\langle \delta T(\hat{\Omega})^2 \delta T(\hat{\Omega}')^2\rangle_c$. Lensing studies involving the CMB can achieve higher signal-to-noise ratio at the level of the bispectrum if we use external data sets to act as a tracers of large-scale structure. However, the lowest order at which the internal detection of CMB lensing is possible is the trispectrum. There have been some attempts to detect non-gaussianity at the level of the trispectrum using e.g. COBE 4yr data release (Kunz et. al. 2001), BOOMERanG data (Troia et. al. 2003) and more recently using WMAP 3 year data (Spergel et al. 2007).

The layout of this paper is as follows. The section §2 we introduce the concept of the higher-order cumulant correlators and how they are linked to corresponding correlation functions in real space. In §3 we discuss the harmonic transforms of the cumulant correlators and their relations to the corresponding multi-spectra. We also discuss estimators based on pseudo-C_l (PCL) estimators used for power spectrum estimation and generalise them to two-point cumulant correlators at higher order in this section. In §4 we briefly discuss the "*local*" models for initial perturbations and the resulting trispectrum. These models are then used in §4 to optimise the power spectra associated with the multi-spectra. This approach is nearly optimal, describing the mode-mode coupling using the "*fraction of sky*" approach familiar from other studies. In §5 we present an estimator which is nearly optimal and can take into account partial sky coverage as well as realistic inhomogeneous noise resulting from the scanning strategy. This estimator can work directly with any specific theoretical model for primordial trispectra, based on concepts of matched filtering, and generalises the results obtained previously by Munshi & Heavens (2009). We present results both for one-point estimators and also generalise them to two corresponding power spectra. §6 is devoted to adding relevant linear correction terms to these estimators in the absence of spherical symmetry. Next, in §7 we introduce inverse covariance weighting and design a trispectrum estimator which is optimal for arbitrary scanning strategy. The estimator used in this section will also be useful also in estimating secondary non-Gaussianity. For homogeneous noise and all-sky coverage the estimators are identical. Finally, §8 is devoted to concluding discussions and future prospects.

2 CORRELATION FUNCTIONS AND THE CUMULANT CORRELATORS

The temperature fluctuations of the Cosmic Microwave Background (CMB) are typically assumed to be a realisation of statistically isotropic Gaussian random field. For Gaussian perturbations all the information needed to provide a complete statistical description is contained in the power spectrum of the distribution; for a non-Gaussian distribution higher-order correlation functions are also needed. With the assumption of *weak* non-Gaussianity only the first few correlation functions are needed to describe the departure from Gaussianity. We will denote the n-point correlation function by $\xi_N(\hat\Omega_1,\dots,\hat\Omega_n)$. The n-point correlation functions are decomposed into parts which are purely Gaussian in nature and those which signify departures from Gaussianity. These are also known as *connected* and *disconnected* terms because of their representation by respective diagrams; see Bernardeau et al. (2002) for more details. At the level of the four-point correlation function the corresponding connected part, denoted by the subscript $\langle\dots\rangle_c$), μ_4 can be defined as:

$$\mu_4(\hat\Omega_1,\dots,\hat\Omega_4) = \langle \delta T(\hat\Omega_1)\dots\delta T(\hat\Omega_4)\rangle_c. \tag{1}$$

The connected component of the four-point function will be exactly zero for a purely Gaussian temperature field. The Gaussian contribution on the other hand can be written as a product of two two-point correlation functions. So the total four-point correlation function can be written as the sum of the connected and the disconnected part:

$$\xi_4(\hat\Omega_1\dots\hat\Omega_4) = \xi_2(\hat\Omega_1,\hat\Omega_2)\xi_2(\hat\Omega_3,\hat\Omega_4) + \xi_2(\hat\Omega_1,\hat\Omega_3)\xi_2(\hat\Omega_2,\hat\Omega_4) + \xi_2(\hat\Omega_1,\hat\Omega_4))\xi_2(\hat\Omega_2,\hat\Omega_3) + \mu_4(\hat\Omega_1\dots\hat\Omega_4). \tag{2}$$

As we will see, the Gaussian part will add to the scatter associated with any estimator for the four-point correlation function. At lower level $\xi_2(\hat\Omega_1,\hat\Omega_2) = \mu_2(\hat\Omega_1,\hat\Omega_2)$ and $\xi_3(\hat\Omega_1\dots\hat\Omega_3) = \mu_3(\hat\Omega_1\dots\hat\Omega_3)$ and hence there are no disconnected parts. The number of degrees of freedom associated with higher-order correlations increase exponentially with its order. This is mainly due to the increased number of configurations possible for which one can measure a higher-order correlation function. The cumulant correlators are defined by identifying all available vertices or points to just two points. There are two such cumulant correlators which can constructed at the four-point level.

$$\xi_{31} \equiv \langle \delta^3 T(\hat\Omega_1)\delta T(\hat\Omega_2)\rangle_c; \qquad\qquad \xi_{22} \equiv \langle \delta^2 T(\hat\Omega_1)\delta T^2(\hat\Omega_2)\rangle_c. \tag{3}$$

These two degenerate sets of cumulant correlators carry information at the level of four-point, but they are essentially two-point correlation functions and can be studied in the harmonic domain by their associated power spectra. The first, ξ_{31} was studied in the context of 21cm surveys by Cooray (2006) and Cooray, Li & Melchiorri (2008) and the second ξ_{22}, was shown to have the power to separate the lensing contribution from the Kinetic Sunyaév Zeldovich effect (Riquelmel & Spergel 2006). These are natural generalisations of their third-order counterpart recently studied by Munshi & Heavens (2009), who introduced their optimised form for direct use on realistic data. These were later used by Smidt et al. (2009) to estimate f_{NL} and Calabrese et al. (2009) to study lensing-secondary correlations from WMAP5 data.

Although there are various advantages of working in real space, it is often easier to work in the harmonic domain. The main motivation to work in the harmonic domain is linked to the fact that inflationary models predict a well-defined peak structure for the power spectrum. These structures are well-known diagnostics for constraining cosmology at various levels. This is true for higher-order multi-spectra as they also involve the effect of the transfer functions. Note that the noise in CMB experiments is typically assumed to be Gaussian and will therefore contribute only to the disconnected terms.

3 FOURIER TRANSFORMS OF CUMULANT CORRELATORS AND THEIR OPTIMUM ESTIMATORS

The real-space correlation functions are clearly very important tools which can be used in surveys with patchy sky coverage. However recent CMB surveys scan the sky with near all-sky coverage. This makes a harmonic-space description more appropriate as various symmetries can be included in a more straightforward way. For Gaussian random fields with all-sky coverage the estimates of various statistics are loosely speaking uncorrelated. Even for a non-Gaussian field, they are reasonably uncorrelated at different angular scales or at different l as long as the non-Gaussianity is weak. We begin by introducing the harmonic transform of the observed temperature map $\delta T(\hat\Omega)$ ($\hat\Omega = (\theta,\phi)$) for all-sky coverage:

$$a_{lm} \equiv \int d\hat\Omega \frac{\Delta T(\hat\Omega)}{T} Y^*_{lm}(\hat\Omega) \equiv \int d\hat\Omega \delta T(\hat\Omega) Y^*_{lm}(\hat\Omega). \tag{4}$$

Realistically however, we will only be observing the part of the sky which is not masked by Galactic foregrounds. The window function $w(\Omega)$ which we will take as a completely general window can be used to define what is known as the pseudo harmonics which we designate as $\tilde a_{lm}$.

$$\tilde a_{lm} \equiv \int d\hat\Omega w(\hat\Omega)\delta T(\hat\Omega) Y^*_{lm}(\hat\Omega). \tag{5}$$

We follow this description for rest of this paper. Any statistic X obtained from the masked sky will be denoted as $\tilde X$ and the estimated all-sky version will be denoted $\hat X$. The ensemble averages of the unbiased all-sky estimators which coincide with the theoretical models will be denoted only by the corresponding latin symbol X.

3.1 Two-Point Estimators for the Trispectrum $C_l^{(3,1)}$, $C_l^{(2,2)}$

At the level of four point cumulant correlators we have two different estimators which can independently be used to study the trispectrum. These estimators that we discuss here are direct harmonics transforms of these two point correlators. We introduced the cumulant correlators in Eq.(3). The two estimators we define in this section are related to $\langle \delta T^2(\hat\Omega)\delta^2 T(\hat\Omega')\rangle$ and $\langle \delta T^3(\hat\Omega)\delta T(\hat\Omega')\rangle$ through harmonic transforms. For the cubic function $\delta T^3(\hat\Omega)$, we denote the harmonic transform as $a^{(3)}_{lm}$ and similarly $a^{(2)}_{lm}$ is the harmonic transform of the quadratic form $\delta T^2(\hat\Omega)$. We obtain the following relations for all-sky coverage, and for the pseudo-harmonics $\tilde a^{(2)}_{lm}$ with a mask.

$$a_{lm}^{(2)} = \sum_{l_1 m_1} \sum_{l_2 m_2} a_{l_1 m_1} a_{l_2 m_2} \int d\hat{\Omega} Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{lm}^*(\hat{\Omega}); \tag{6}$$

$$\tilde{a}_{lm}^{(2)} = \sum_{l_1 m_1} \dots \sum_{l_3 m_3} a_{l_1 m_1} a_{l_2 m_2} w_{l_3 m_3} \int d\hat{\Omega} Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{l_3 m_3}(\hat{\Omega}) Y^*_{lm}(\hat{\Omega}) = \sum_{(l'm')} K_{lml'm'} a_{l'm'}^{(2)}. \tag{7}$$

The corresponding results for $a_{lm}^{(3)}$ and $\tilde{a}_{lm}^{(3)}$ are as follows:

$$a_{lm}^{(3)} = \sum_{l_1 m_1} \dots \sum_{l_3 m_3} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \int d\hat{\Omega} Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{l_3 m_3}(\hat{\Omega}) Y^*_{lm}(\hat{\Omega}); \tag{8}$$

$$\tilde{a}_{lm}^{(3)} = \sum_{l_1 m_1} \dots \sum_{l_4 m_4} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m\ 3} w_{l_4 m_4} \int d\hat{\Omega} Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) Y_{l_3 m_3}(\hat{\Omega}) Y_{l_4 m_4}(\hat{\Omega}) Y^*_{lm}(\hat{\Omega}) = \sum_{(l'm')} K_{lml'm'} a_{l'm'}^{(3)}. \tag{9}$$

The coupling matrix $K_{lml'm'}$ encodes information about the mode coupling which is introduced because of the masking of the sky (Hivon et al. 2001):

$$K_{l_1 m_1 l_2 m_2}[w] = \int w(\hat{\Omega}) Y_{l_1 m_1}(\hat{\Omega}) Y_{l_2 m_2}(\hat{\Omega}) \mathrm{d}\hat{\Omega} = \sum_{l_3 m_3} \tilde{w}_{l_3 m_3} \left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi} \right)^{1/2} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right). \tag{10}$$

The matrices here denote the 3J symbols (Edmonds 1968). Using the harmonic transforms we define the following power spectra which can directly probe the trispectra.

$$\mathcal{C}_l^{(2,2)} = \frac{1}{2l+1} \sum_m a_{lm}^{(2)*} a_{lm}^{(2)}; \quad \tilde{\mathcal{C}}_l^{(2,2)} = \frac{1}{2l+1} \sum_m \tilde{a}_{lm}^{(2)*} \tilde{a}_{lm}^{(2)}. \tag{11}$$

From the consideration of isotropy and homogeneity we can write the following relations:

$$\langle a_{lm}^{(2)} a_{lm}^{(2)*} \rangle_c = \mathcal{C}_l^{(2,2)} \delta_{ll'} \delta_{mm'}; \qquad \langle a_{lm}^{(3)} a_{lm}^{*} \rangle_c = \mathcal{C}_l^{(3,1)} \delta_{ll'} \delta_{mm'}. \tag{12}$$

The pseudo-power spectrum which is recovered from the masked harmonics is defined in an analogous way. Finally the resulting power spectra $C_l^{(2,2)}$ can be expressed in terms of the trispectrum $T^{l_1 l_2}_{l_3 l_4}(l)$ by the following expression:

$$\mathcal{C}_l^{(2,2)} = \sum_{l_1 l_2 l_3 l_4} T^{l_3 l_4}_{l_1 l_2}(l) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \sqrt{\frac{(2l_3+1)(2l_4+1)}{4\pi(2l+1)}} \left(\begin{array}{ccc} l_1 & l_2 & l \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{ccc} l_3 & l_4 & l \\ 0 & 0 & 0 \end{array} \right). \tag{13}$$

The trispectrum which is introduced here is a four-point correlation function in harmonic space. The definition here ensures that it is invariant under various transformations; see Hu (2000) and Hu & Okamoto (2002) for detailed discussion:

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \rangle_c = \sum_{LM} (-1)^M T^{l_3 l_4}_{l_1 l_2}(L) \left(\begin{array}{ccc} l_1 & l_2 & L \\ m_1 & m_2 & M \end{array} \right) \left(\begin{array}{ccc} l_3 & l_4 & L \\ m_3 & m_4 & -M \end{array} \right). \tag{14}$$

Partial sky coverage can be dealt with in an exactly similar manner. By using the harmonic transforms of the masked sky and expressing the masked harmonics in terms of all-sky harmonics we can relate the PCL versions of these power spectra $\tilde{C}_l^{(2,2)}$ in terms of its all-sky components $C_l^{(2,2)}$. This involves the harmonic transform of the mask W_{lm} and its associated power spectrum W_l. The coupling matrix for the $M_{ll'}$ is the same as that which is used for inverting the PCL power spectrum to recover the unbiased power spectrum.

$$\begin{aligned} \tilde{\mathcal{C}}_l^{(2,2)} &= \sum_{l'l''} (2l'+1) \left(\begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right)^2 \frac{(2l''+1)}{4\pi} |w^2_{l''}| \\ &\times \sum_{l_1 l_2 l_3 L} T^{l_1 l_2}_{l_3 l_4}(l') \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l'+1)}} \sqrt{\frac{(2l_3+1)(2l_4+1)}{4\pi(2l'+1)}} \left(\begin{array}{ccc} l_1 & l_2 & l' \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{ccc} l_3 & l_4 & l' \\ 0 & 0 & 0 \end{array} \right) = \sum_{l'} M_{ll'} \mathcal{C}_{l'}^{(2,2)}. \end{aligned} \tag{15}$$

where

$$M_{ll'} = (2l'+1) \sum_{l''} \left(\begin{array}{ccc} l & l' & l'' \\ 0 & 0 & 0 \end{array} \right)^2 \frac{(2l''+1)}{4\pi} |w^2_{l''}|. \tag{16}$$

In the literature the power spectrum $c_l^{(2,2)}$ is also referred as the *second spectrum*. The other degenerate cumulant correlator of the same order that contains information about the trispectrum can be written as:

$$\mathcal{C}_l^{(3,1)} = \frac{1}{2l+1} \sum_m \mathrm{Real} \left\{ a_{lm}^{(3)*} a_{lm}^{(1)} \right\}; \quad \tilde{\mathcal{C}}_l^{(3,1)} = \frac{1}{2l+1} \sum_m \mathrm{Real} \left\{ \tilde{a}_{lm}^{(3)*} \tilde{a}_{lm}^{(1)} \right\}. \tag{17}$$

The resulting power spectrum which probes the various components of the trispectrum with different weights can be expressed as follows:

$$\mathcal{C}_l^{(3,1)} = \sum_{l_1 l_2 l_3 L} T^{l_1 l_2}_{l_3 l}(L) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi\ (2L+1)}} \sqrt{\frac{(2L+1)(2l_3+1)}{4\pi(2l+1)}} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l_3 & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

With partial sky coverage we can proceed as before to connect the PCL version of the estimator $\mathcal{C}_l^{(3,1)}$ to its all-sky analogue. The resulting estimators combine the values of various components of the trispectrum with different weighting. The associated power-spectrum will therefore have a different dependence on parameters describing the primordial trispectrum.

$$\begin{aligned} \mathcal{C}_l^{(3,1)} &= \sum_{l'l''} (2l'+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{(2l''+1)}{4\pi} |w^2_{l''}| \\ &\times \sum_{l_1 l_2 l_3 L} T^{l_1 l_2}_{l_3 l}(L) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}} \sqrt{\frac{(2L+1)(2l_3+1)}{4\pi(2l'+1)}} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l_3 & l' \\ 0 & 0 & 0 \end{pmatrix} = \sum_{l'} M_{ll'} \mathcal{C}_{l'}^{(3,1)}. \end{aligned} \tag{19}$$

The links to real space cumulant correlators are same as their third order counterpart:

$$\langle \delta^2 T(\hat\Omega) \delta^2 T(\hat\Omega') \rangle_c = \frac{1}{4\pi} \sum_l (2l+1) P_l(\cos(\hat\Omega\cdot\hat\Omega')) \mathcal{C}_l^{(2,2)}; \qquad \langle \delta^3 T(\hat\Omega) \delta T(\hat\Omega') \rangle_c = \frac{1}{4\pi} \sum_l (2l+1) P_l(\cos(\hat\Omega\cdot\hat\Omega')) \mathcal{C}_l^{(3,1)}. \tag{20}$$

Hence we can conclude that it is possible to generalise these results to an arbitrary mask with arbitrary weighting functions. The deconvolved set of estimators at order p,q can be written as follows:

$$\hat C_l^{p,q} = M^{-1}_{ll'} \tilde C_{l'}^{p,q}. \tag{21}$$

This is one of the important results of this paper. The mask used for various orders p,q is the same, but with the increasing order the number of degenerate power-spectra that can be constructed from a multi-spectrum increases drastically. As we have seen at the level of bispectrum we can keep one of the triangle sides fixed and sum over all contributions from all possible configurations of the triangle. Similarly for the trispectrum we can keep one of the sides of the rectangle fixed, or one of the diagonals fixed, and sum over all possible configurations. The possibilities increase as we move to higher-order in multispectra. Another related complication would be from the number of disconnected terms which need to be subtracted as they simply correspond to Gaussian contributions. At the 4-point level we need to subtract the disconnected pieces from the dominant Gaussian component of temperature fluctuations including the noise (Hu 2000; Hu & Okamoto 2002):

$$G^{l_1 l_2}_{l_3 l_4}(L) = (-1)^{l_1+l_3} \sqrt{(2l_1+1)(2l_3+1)} C_{l_1} C_{l_3} \delta_{L0} \delta_{l_1 l_2} \delta_{l_2 l_3} + (2L+1) C_{l_1} C_{l_2} \left[(-1)^{l_2+l_3+L} \delta_{l_1 l_3} \delta_{l_2 l_4} + \delta_{l_1 l_4} \delta_{l_2 l_3} \right], \tag{22}$$

where C_l is the power spectrum including noise.

For the all-sky case and if we restrict only to modes with ordering $l_1 \le l_2 \le l_3 \le l_4$, the non-zero component corresponds to terms with $L=0$ or $l_1 = l_2 = l_3 = l_4$. However for arbitrary sky coverage, which results in mode-mode coupling, no such general comments can be made. Estimators developed here in their all-sky form are known in the literature, and the results here show how to generalise them for partial sky coverage. However the main interest in data compression lies in using optimal weights for the compression, which is model-dependent as the weights depend on the specific model being probed.

If we introduce the Gaussian contributions to $C_l^{(3,1)}$ as $G_l^{(3,1)}$, and to $C_l^{(2,2)}$ as $G_l^{(2,2)}$, in the presence of partial sky coverage we can write:

$$\tilde G_l^{(3,1)} = \sum_{l'} M_{ll'} G_{l'}^{(3,1)}; \qquad \tilde G_l^{(2,2)} = \sum_{l'} M_{ll'} G_{l'}^{(2,2)}. \tag{23}$$

For all-sky coverage one can obtain the corresponding results by replacing $T^{l_1,l_2}_{l_3,l_4}(L)$ by $G^{l_1,l_2}_{l_3,l_4}(L)$ in respective equations:

$$G_l^{(3,1)} = \sum_{l_1 l_2 l_3 L} G^{l_1,l_2}_{l_3,l}(L) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi\ (2L+1)}} \sqrt{\frac{(2L+1)(2l_3+1)}{4\pi(2l+1)}} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & l_3 & l \\ 0 & 0 & 0 \end{pmatrix} \tag{24}$$

$$G_l^{(2,2)} = \sum_{l_1 l_2 l_3 L} G^{l_3,l_4}_{l_1,l_2}(l) \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \sqrt{\frac{(2l_3+1)(2l_4+1)}{4\pi(2l+1)}} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

To subtract the disconnected Gaussian part, we use simulations. These are constructed from the same power spectrum that describes the non-Gaussian maps. Noise realisations which describe the actual map are also introduced in the Gaussian maps and exactly the same mask is used. This will ensure that the estimator remains unbiased. If we denote the final estimators after the subtraction of Gaussian disconnected parts by $D_l^{3,1}$ and $D_l^{(2,2)}$ we can write for all-sky coverage:

$$\hat D_l^{(2,2)} \equiv \hat C_l^{(2,2)} - \hat G_l^{(2,2)}; \qquad \hat D_l^{(3,1)} \equiv \hat C_l^{(3,1)} - \hat G_l^{(3,1)}. \tag{26}$$

In the presence of a mask, deconvolved estimators are related to the masked estimators by the mixing matrix $M_{ll'}$:

$$\hat D_l^{(2,2)} = \sum_{ll'} M^{-1}_{ll'} (\tilde C_{l'}^{(2,2)} - \tilde G_{l'}^{(2,2)}); \qquad \hat D_l^{(3,1)} = \sum_{ll'} M^{-1}_{ll'} (\tilde C_{l'}^{(3,1)} - \tilde G_{l'}^{(3,1)}). \tag{27}$$

Practical implementation of these estimators can provide a quick sanity check of a pipeline design for non-Gaussian estimators. While it is useful to keep in mind that these estimators are sub-optimal they are nevertheless *unbiased* and, depending on various choices of f_{NL} and g_{NL}, they can provide an analytical basis for computation of the scatter and cross-correlation among various estimators associated with different levels of the correlation hierarchy.

The dependence of $D_l^{(2,2)}$ and $D_l^{(3,1)}$ on f_{NL} and g_{NL} are different and hence can be used to provide independent constraints on both of these parameters without using third-order estimators and might be useful for providing cross-checks.

3.2 One-Point Unoptimised Estimators for the Trispectrum $C_l^{(4)}$

The one-point cumulants at third order can be written in terms of the bispectra as:

$$\mu_3 = \langle \delta T^3(\hat{\Omega}) \rangle = \frac{1}{4\pi} \int \delta T^3(\hat{\Omega}) d\hat{\Omega} = \frac{1}{4\pi} \sum_{l_1 l_2 l_3} h_{l_1 l_2 l_3} B_{l_1 l_2 l_3}. \tag{28}$$

Similarly the one-point cumulant at fourth order can be written in terms as the trispectra as:

$$\mu_4 = \langle \delta T^4(\hat{\Omega}) \rangle_c = \frac{1}{4\pi} \int \delta T^4(\hat{\Omega}) d\hat{\Omega} = \frac{1}{4\pi} \sum_L \sum_{l_1 l_2 l_3 l_4} h_{l_1 l_2 L} h_{l_3 l_4 L} T^{l_3 l_4}_{l_1 l_2}(L). \tag{29}$$

We can also define the S_N parameters generally used in the literature as:

$$S_3 = \mu_3; \qquad\qquad S_4 = \mu_4 - 3\mu_2^2. \tag{30}$$

Throughout this discussion we have absorbed the experimental beam $b_l = \exp\{-l(l+1)/2\sigma_b^2\}$ in the harmonics a_{lm} unless it is displayed explicitly. Here $\sigma_b = \mathrm{FWHM}/\sqrt{8ln2}$ for a gaussian beam. Alternatively it is also possible to define by respective powers of μ_2 to make these one-point estimators less sensitive to the normalisation of the power spectra. In that case we will have $S_3 = \mu_3/\mu_2^2$. These moments at fourth order are generalisations of the third-order moments, used as a basis for the construction of estimators for $f_{\rm NL}$ by introducing the optimal weights $A(r,\hat{\Omega})$ and $B(r,\hat{\Omega})$.

4 MODELS FOR PRIMORDIAL NON-GAUSSIANITY AND CONSTRUCTION OF OPTIMAL WEIGHTS

The optimisation techniques that we introduce in next section follow the discussion in Munshi & Heavens (2009). The optimisation procedure depends on construction of three-dimensional fields from the harmonic components of the temperature fields a_{lm} with suitable weighting with respective functions which describes primordial non-Gaussianity(Yadav, Komatsu,& Wandelt 2007). These weights make the estimators act optimum and the matched filtering technique adopted ensures that the response to the observed non-Gaussianity is maximum when it matches with primordial non-gaussianity corresponding to the weights.

In the linear regime the curvature perturbations which generate the fluctuations in the CMB sky are written as:

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Phi(\mathbf{k}) \Delta_l^T(k) Y_{lm}(\hat{k}). \tag{31}$$

We will need the following functions to construct the harmonic space trispectra as well as to generate weights for construction of optimal estimators (for a more complete description of predicting trispectrum from a given Inflationary prediction see (Hu 2000; Hu & Okamoto 2002)) :

$$\alpha_l(r) = \frac{2}{\pi}\int_0^\infty k^2 dk \Delta_l(k) j_l(kr); \;\; \beta_l(r) = \frac{2}{\pi}\int_0^\infty k^2 dk P_\phi(k) \Delta_l(k) j_l(kr); \;\; \mu_l(r) = \frac{4}{\pi}\int_0^\infty k^2 dk \Delta_l(k) j_l(kr) \tag{32}$$

In the limit of low multipoles where the perturbations are mainly dominated by Sachs-Wolfe Effect the transfer functions $\Delta_l(k)$ takes a rather simple form $\Delta_l(k) = 1/3 j_l(kr_*)$ where $r_* = (\eta_0 - \eta_{dec})$ denotes the time elapsed between cosmic recombination and the present epoch. In general the transfer function needs to be computed numerically. The *local* Model for the primordial bispectrum and trispectrum can be constructed by going beyond linear theory in the expansion of the $\Phi(x)$. Additional parameters f_{NL} and g_{NL} are introduced which need to be estimated from observation. As discussed in the introduction, g_{NL} can be linked to r the scalar to tensor ratio in a specific inflationary model and hence expected to be small.

$$\Phi(\mathbf{x}) = \Phi_L(\mathbf{x}) + f_{NL}\left(\Phi_L^2(\mathbf{x}) - \langle \Phi_L^2(\mathbf{x})\rangle\right) + g_{NL}\Phi_L^3(\mathbf{x}) + h_{NL}\left(\Phi_L^4(\mathbf{x}) - 3\langle \Phi_L^2(\mathbf{x})\rangle\right) + \dots \tag{33}$$

We will only consider the local model of primordial non-Gaussianity in this paper and only adiabatic initial perturbations. More complicated cases of primordial non-Gaussianity will be dealt with elsewhere. In terms of inflationary potential $V(\phi)$ associated with a scalar potential ϕ one can express these constants as (Hu 2000):

$$f_{\rm NL} = -\frac{5}{6}\frac{1}{8\pi G}\frac{\partial^2 \ln V(\phi)}{\partial\phi^2}; \qquad\qquad g_{\rm NL} = \frac{25}{54}\frac{1}{8\pi G}\left[2\left(\frac{\partial^2 \ln V(\phi)}{\partial\phi^2}\right) - \left(\frac{\partial^3 \ln V(\phi)}{\partial\phi^3}\right)\left(\frac{\partial \ln V(\phi)}{\partial\phi}\right)\right]. \tag{34}$$

There are two contribution to primordial non-Gaussianity. The first part is parametrised by $f_{\rm NL}$ and the second contribution is proportional to a new parameter which appears at fourth order which we denote by $g_{\rm NL}$. From theoretical considerations in generic models of inflation one would expect $g_{\rm NL} \le r/50$ with r being the scalar to tensor ratio (Serry, Lidsey & Sloth 2008).

Following (Hu 2000) we can expand above expression in Fourier space to write:

$$\Phi_2(k) = \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \Phi_L(\mathbf{k}+\mathbf{k}_1)\Phi_L^*(\mathbf{k}_1) - (2\pi)^3\delta_D(\mathbf{k}) \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} P_{\Phi\Phi}(\mathbf{k}_1) \tag{35}$$

$$\Phi_3(k) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Phi_L(\mathbf{k}_1)\Phi_l(\mathbf{k}_2)\Phi_l^*(\mathbf{k}_1). \tag{36}$$

The resulting trispectra associated with these perturbations can be expressed as:

$$T_\Phi(k_1,k_2,k_3,k_4) \equiv \langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\delta(\mathbf{k}_3)\delta(\mathbf{k}_4)\rangle_c = \int \frac{d^3\mathbf{K}}{(2\pi)^3} \delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{K})\delta_D(\mathbf{k}_3+\mathbf{k}_4+\mathbf{K})\mathcal{T}_\Phi(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{K}). \tag{37}$$

where the $\mathcal{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{K})$ can be decomposed into two different constituents.

$$\mathcal{T}_\Phi^{(2)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{K}) = 4f_{NL}^2 P_\phi(\mathbf{K})P_\phi(\mathbf{k}_1)P_\Phi(\mathbf{k}_3) \tag{38}$$

$$\mathcal{T}_\Phi^{(3)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4,\mathbf{K}) = g_{NL}\left\{P_\phi(\mathbf{K})P_\phi(\mathbf{k}_1)P_\Phi(\mathbf{k}_3) + P_\phi(\mathbf{K})P_\phi(\mathbf{k}_1)P_\Phi(\mathbf{k}_3)\right\} \tag{39}$$

The CMB trispectrum now can be written as

$$\begin{aligned} T^{l_1l_2}_{l_3l_4}(L) = \quad & 4f_{\rm NL}^2 h_{l_1l_2L}h_{l_3l_4L}\int r_1^2dr_1r_2^2dr_2F_L(r_1,r_2)\alpha_{l_1}(r_1)\beta_{l_2}(r_1)\alpha_{l_3}(r_2)\beta_{l_4}(r_2) \\ & + g_{\rm NL}h_{l_1l_2L}h_{l_3l_4L}\int r^2dr\beta_{l_2}(r)\beta_{l_4}(r)[\mu_{l_1}(r)\beta_{l_3}(r) + \mu_{l_3}(r)\beta_{l_1}(r)]. \end{aligned} \tag{40}$$

For detailed descriptions involving polarisation maps see (Hu & Okamoto 2002; Hu 2000; Komatsu & Spergel 2001; Kogo et al. 2006). The CMB bispectrum which describes departures from Gaussianity at the lowest level can be similarly written as:

$$B_{l_1l_2l_3} = 2f_{\rm NL}h_{l_1l_2l_3}\int r^2dr\left[\alpha_{l_1}(r)\beta_{l_2}(r)\beta_{l_3}(r) + \alpha_{l_2}(r)\beta_{l_3}(r)\beta_{l_1}(r) + \alpha_{l_3}(r)\beta_{l_2}(r)\beta_{l_1}(r)\right]. \tag{41}$$

We have defined the following form factor to simplify the display:

$$h_{l_1l_2l_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array}\right). \tag{42}$$

The extension to order beyond presented here to involve higher-order Taylor coefficients may not be practically useful as the detector noise and the cosmic variance may prohibit any reasonable signal-to-noise.

5 NEAR OPTIMAL ESTIMATORS FOR TRISPECTRUM

We will develop two estimators for extraction of power spectra associated with the trispectrum (Hu 2000; Hu & Okamoto 2002) in this section $\mathcal{K}_l^{3,1)}$ and $\mathcal{K}_l^{2,2}$. These estimators are optimised version of similar estimators considered in the previous section. The derivation here parallels that of the previous section where unoptimised versions of these estimators were developed which uses the PCL type estimators. We will start by introducing four different fields which are constructed from the temperature fields. These fields are defined over the observed part of the sky and and are constructed using suitable weights to temperature harmonics. These weights are functions of radial distance $\alpha_l(r), \beta_l(r), \mu_l(r)$ so the constructed fields are 3D fields. This method follows the same technique as introduced by Komatsu, Spergel & Wandelt (2005).

$$A(r,\hat\Omega) = \sum_{lm} A_{lm}Y_{lm}(\hat\Omega); \qquad B(r,\hat\Omega) = \sum_{lm} B_{lm}Y_{lm}(\hat\Omega); \qquad M(r,\hat\Omega) = \sum_{lm} M_{lm}Y_{lm}(\hat\Omega); \tag{43}$$

$$A_{lm} = \frac{\alpha_{lm}}{C_l}a_{lm}; \;\; B_{lm} = \frac{\beta_{lm}}{C_l}a_{lm}; \;\; M_{lm} = \frac{\mu_{lm}}{C_l}a_{lm}. \tag{44}$$

We have absorbed the beam smoothing into the corresponding harmonic coefficients. In addition to the weighting functions we will also need to define overlap integrals $F_L(r_1,r_2)$ which act as a kernel for cross-correlating fields at two different radial distances. Note that the overlap integral depends on the quantum number L:

$$F_L(r_1,r_2) = \frac{2}{\pi}\int k^2dkP_{\Phi\Phi}(k)j_L(kr_1)j_L(kr_2). \tag{45}$$

In the following subsections we will use specific forms for the trispectra to construct optimal estimator. Although the estimators developed here are specific to a given model clearly for any given model for the trispectra we can obtain similar construction in an optimal way. The weighting of these harmonics make the estimator a match-filtering one which ensures maximum response when the trispectra obtained from the data matches with that with theoretical construct. Inverse variance weighting ensures that the estimator remains optimal for a specific survey strategy.

5.1 Estimator for $\mathcal{K}^{(4)}$

The one-point estimators are simpler though they compress all available information in one number. One-point estimators can also be computed directly in real space and hence are simpler to compute when dealing with partial sky coverage in the presence of inhomogeneous noise. Using notations introduced above, we can write the one-point cumulant at fourth order by the following expression:

$$\mathcal{K}^{(4)} = 4f_{\rm Nl}^2 \int r_1^2 \int r_2^2 dr_2 F(r_1,r_2) \int d\hat\Omega A(r_1,\hat\Omega)B(r_1,\hat\Omega)B(r_2,\hat\Omega)A(r_2,\hat\Omega) + 2g_{\rm Nl}\int r^2 dr \int d\hat\Omega A(r,\hat\Omega)B^2(r,\hat\Omega)M(r,\hat\Omega) \tag{46}$$

which can be written in a compact form

$$\mathcal{K}^{(4)} = 4f_{\rm NL}^2 \int r_1^2 \int r_2^2 dr_2 F(r_1,r_2)\langle A(r_1,\hat\Omega)B(r_1,\hat\Omega)A(r_2,\hat\Omega)B(r_2,\hat\Omega)\rangle + 2g_{\rm Nl}\int r^2 dr \langle A(r,\hat\Omega)B^2(r,\hat\Omega)M(r,\hat\Omega)\rangle. \tag{47}$$

As expected there are two parts in the contribution. The first part depends on two radial directions through the overlap integral $F(r_1,r_2)$; the second is much simpler and just contains one line-of-sight integration. The amplitude of the first term depends on $f_{\rm NL}^2$ and the second term is proportional to $g_{\rm NL}$. Typically they contribute equally to the resulting estimate.

5.2 Estimator for $\mathcal{K}_l^{(3,1)}$

Moving beyond the one-point cumulants we can construct the estimators of the two power spectra which we discussed before, $C_l^{(3,1)}$ and $C_l^{(2,2)}$. The corresponding optimised versions can be constructed by cross-correlating the fields $A(r_1,\hat\Omega)B(r_1,\hat\Omega)B(r_2,\hat\Omega)$ with $B(r_2,\hat\Omega)$. Clearly in the first case the harmonics depend on two radial distances r_1, r_2 for any given angular direction:

$$A(r_1)B(r_1)B(r_2)|_{lm} = \int A(r_1,\hat\Omega)B(r_1,\hat\Omega)B(r_2,\hat\Omega)\, Y_{lm}(\hat\Omega)\, d\hat\Omega; \qquad A(r_2)|_{lm} = \int A(r_2,\hat\Omega)\, Y_{lm}(\hat\Omega)\, d\hat\Omega. \tag{48}$$

Next we compute the cross-power spectra $\mathcal{J}_l^{AB^2,A}(r_1,r_2)$ which also depend on both radial distances r_1 and r_2:

$$\mathcal{J}_l^{AB^2,A}(r_1,r_2) = \frac{1}{2l+1}\sum_m F_L(r_1,r_2){\rm Real}\left[\{A(r_1)B(r_1)B(r_2)\}_{lm}\{A(r_2)\}^*_{lm}\right]. \tag{49}$$

The construction for the second term is very similar. We start by decomposing the real space product $A(r,\hat\Omega)B^2(r,\hat\Omega)$ and $M(r,\hat\Omega)$ in harmonic space. There is only one radial distance involved in both of these terms.

$$A(r)B^2(r)|_{lm} = \int [A(r,\hat\Omega)B^2(r,\hat\Omega)]\, Y_{lm}(\hat\Omega)\, d\hat\Omega; \qquad M(r,\hat\Omega)|_{lm} = \int M(r) Y_{lm}(\hat\Omega)\, d\hat\Omega. \tag{50}$$

Finally the line-of-sight integral which involves two overlapping contribution through the weighting kernels for the first term and only one for the second gives us the required estimator:

$$\mathcal{K}_l^{(3,1)} = 4f_{\rm nl}^2 \int r_1^2 dr_1 \int r_2^2 dr_2 \mathcal{J}_l^{AB^2,A}(r_1,r_2) + 2g_{\rm nl}\int r^2 dr \mathcal{L}_l^{AB^2,M}(r) \tag{51}$$

Next we show that the construction described above does reduces to an optimum estimator for the power spectrum associated with the trispectrum. The harmonics associated with the product field $A(r_1)B(r_1)B(r_2)$ can be expressed in terms of the functions $\alpha(r)$ and $\beta(r)$:

$$A(r_1)B(r_1)B(r_2)|_{lm} = \sum_{LM}(-1)^M \sum_{lm,l_im_i} a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\ \alpha_{l_1}(r_1)\beta_{l_2}(r_1)\beta_{l_3}(r_2)\mathcal{G}^{m_1m_2M}_{l_1l_2L}\mathcal{G}^{Mm_3m}_{Ll_3l}. \tag{52}$$

The cross-power spectra $\mathcal{J}_l^{AB^2,A}(r_1,r_2)$ can be simplified in terms of the following expression:

$$\mathcal{J}_l^{AB^2,A}(r_1,r_2) = \frac{1}{2l+1}\sum_{LM}(-1)^M\sum_m \{F_L(r_1,r_2)\alpha_{l_1}(r_1)\alpha_{l_2}(r_1)\alpha_{l_2}(r_2)\beta_l(r_2)\}\,\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{lm}\rangle \mathcal{G}^{m_1m_2M}_{l_1l_2L}\mathcal{G}^{m_3mM}_{l_3lL}. \tag{53}$$

Where the Gaunt integral describing the integral involving three spherical harmonics is defined as follows:

$$\mathcal{G}^{m_1m_2m_3}_{l_1l_2l_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}. \tag{54}$$

The second terms can be treated in an analogus way and takes the following form:

$$\mathcal{L}_l^{AB^2,M}(r) = \frac{1}{2l+1}\sum_{LM}(-1)^M\sum_m \{\alpha_{l_1}(r_1)\alpha_{l_2}(r_1)\alpha_{l_2}(r_2)\mu_l(r_2)\}\,\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{lm}\rangle \mathcal{G}^{l_1l_2L}_{l_1l_2L}\mathcal{G}^{l_3lL}_{l_3lL}. \tag{55}$$

Finally when combined these terms as in Eq.(51), we recover the following expression:

$$\mathcal{K}_l^{(3,1)} = \frac{1}{2l+1}\sum_{l_1l_2l_3}\sum_L \frac{1}{\mathcal{C}_{l_1}\mathcal{C}_{l_2}\mathcal{C}_{l_3}\mathcal{C}_l} T^{l_1l_2}_{l_3l}[L]T^{l_1l_2}_{l_3l}[L]. \tag{56}$$

The estimator $\mathcal{K}_l^{(3,1)}$ depends linearly both on $f_{\rm NL}^2$ and $g_{\rm NL}$. In principle, we can use the estimate of $f_{\rm NL}$ from a bispectrum analysis as a prior, or we can use the estimators $\mathcal{S}_l^{(2,1)}$, $\mathcal{K}_l^{(3,1)}$ and $\mathcal{K}_l^{(3,1)}$ to put joint constraints on f_{NL} and g_{NL}. Computational evaluation of either of the power spectra clearly will be more involved as a double integral corresponding to two radial directions needs to be evaluated. Given the low signal-to-noise associated with these power spectra, binning will be essential.

5.3 Estimator for $\mathcal{K}_l^{(2,2)}$

In an analogous way the other power spectra associated with the trispectra can be optimised by the following construction. We start by taking the harmonic transform of the product field $A(r,\hat\Omega)B(r,\hat\Omega)$ evaluated at the same line-of-sight distance r:

$$A(r,\hat\Omega)B(r,\hat\Omega)|_{lm} = \int A(r)B(r)\,Y_{lm}(\hat\Omega)\,d\hat\Omega; \qquad B(r,\hat\Omega)M(r,\hat\Omega)|_{lm} = \int B(r)M(r)\,Y_{lm}(\hat\Omega)\,d\hat\Omega, \tag{57}$$

and contract it with its counterpart at a different distance. The corresponding power spectrum (which is a function of these two line-of-sight distances r_1 and r_2) has a first term

$$\mathcal{J}_l^{AB,AB}(r_1,r_2) = \frac{1}{2l+1}\sum_m\sum_L F_L(r_1,r_2)A(r_1,\hat\Omega)B(r_1,\hat\Omega)|_{lm}A(r_2,\hat\Omega)B(r_2,\hat\Omega)|_{lm}^* \tag{58}$$

Similarly, the second part of the contribution can be constructed by cross-correlating the product of 3D fields $A(\hat\Omega,r_1)B(\hat\Omega,r_1)$ and $B(\hat\Omega,r_2)M(\hat\Omega,r_2)$ evaluated at different r_1 and r_2, with corresponding weighting function $F_L(r_1,r_2)$:

$$\mathcal{L}_l^{AB,BM}(r) = \frac{1}{2l+1}\sum_m A(r,\hat\Omega)B(r,\hat\Omega)|_{lm}B(r,\hat\Omega)M(r,\hat\Omega)|_{lm}^* \tag{59}$$

Finally, the estimator is constructed as:

$$\mathcal{K}_l^{(2,2)} = 4f_{nl}^2\int r_1^2dr_1\int r_2^2dr_2\mathcal{J}_l^{AB,AB}(r_1,r_2) + 2g_{nl}\int r^2dr\mathcal{L}_l^{AB,BM}(r) \tag{60}$$

To see they do correspond to an optimum estimator we use the harmonic expansions and follow the same procedure outlined before:

$$\mathcal{L}_l^{AB,BM}(r) = \frac{1}{2l+1}\sum_m(-1)^m\sum_{l_im_i}\{\alpha_{l_1}(r)\alpha_{l_2}(r)\beta_{l_3}(r)\mu_{l_4}(r)\}\,\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4}\rangle\mathcal{G}_{l_1l_2l}^{m_1m_2m}\mathcal{G}_{l_3l_4l}^{m_3m_4m}. \tag{61}$$

$$\mathcal{J}_l^{AB,AB}(r_1,r_2) = \frac{1}{2l+1}\sum_{LM}(-1)^M\sum_m\{F_L(r_1,r_2)\alpha_{l_1}(r_1)\beta_{l_2}(r_1)\alpha_{l_2}(r_2)\beta_l(r_2)\}\,\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{lm}\rangle\mathcal{G}_{l_1l_2L}^{m_1m_2M}\mathcal{G}_{l_3l_4L}^{m_3m_4M}. \tag{62}$$

Here we notice that $\mathcal{J}_l^{AB,AB}(r_1,r_2)$ is invariant under exchange of r_1 and r_2 but $\mathcal{J}_l^{AB,BM}(r_1,r_2)$ is not. Finally, joining the various contributions to construct the final estimator, as given in Eq.(60), which involves a line-of-sight integration:

$$\mathcal{K}_l^{(2,2)} = \frac{1}{2l+1}\sum_{l_im_i}\frac{1}{\mathcal{C}_{l1}\mathcal{C}_{l_2}\mathcal{C}_{l_3}\mathcal{C}_{l_4}}T_{l_3l_4}^{l_1l_2}(l)T_{l_3l_4}^{l_1l_2}(l). \tag{63}$$

The prefactors associated with $f_{\rm NL}^2$ and $g_{\rm NL}$ are different in the linear combinations $\mathcal{K}_l^{(2,2)}$ and $\mathcal{K}_l^{(3,1)}$, and hence even without using information from third-order we can estimate both from fourth order alone.

Figs. 1 and 2 shows the primordial $\mathcal{K}_l^{(2,2)}$ and $\mathcal{K}_l^{(3,2)}$ for the WMAP5 best-fit cosmological parameters (Dunkley et al. 2009), integrated over a range of 500 Mpc around recombination, and shown as a function of harmonic number l. The computations of $\mathcal{K}_l$ typically scales as l_{max}^3 with resolution which makes it quite expensive for high resolution studies. The computations are largely dominated by computations of 3J symbols. For a given configuration number of non-zero 3J symbols for which the computations are required roughly scales as l_{max}^3 which explains the scaling. For more details about noise and beam see Smidt et al. (2009)

The unbiased version of $\mathcal{K}_l^{(2,2)}$ has been used in the context of study of lensing effects on CMB maps (Cooray & Kesden 2003). While cross-correlational analysis can be helpful for detection of lensing on CMB maps for internal detection involving only CMB maps an analysis at the trispectrum level is necessary.

5.4 Linking various Estimators

The one-point estimators can be expressed as a sum over all ls of the estimators $\mathcal{K}_l^{(3,1)}$ or $\mathcal{K}_l^{(2,2)}$. Therefore the corresponding optimised one-point estimators $\mathcal{K}^{(4)}$ can be written as:

$$\mathcal{K}^{(4)} = \sum_{l_iL}\frac{T_{l_1l_2}^{l_3l_4}(L)\hat{T}_{l_1l_2}^{l_3l_4}(L)}{C_{l_1}C_{l_2}C_{l_3}C_{l_4}}. \tag{64}$$

Here $\hat{T}_{l_1l_2}^{l_3l_4}(L)$ denotes the direct estimator from the harmonic transforms and $T_{l_1l_2}^{l_3l_4}(L)$ act as weights which are constructed using the optimisation function discussed above. Similarly, at the two-point level the corresponding power spectra can be written as:

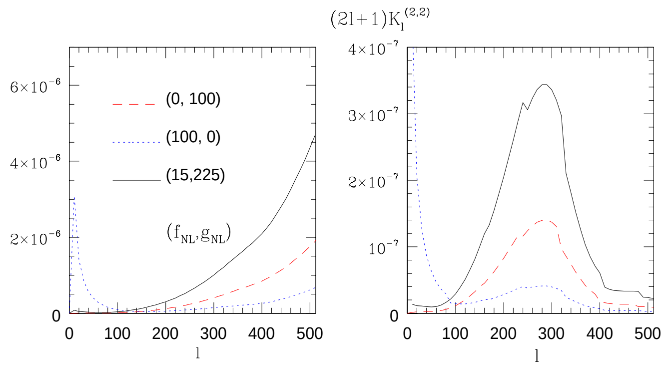

Figure 1. The trispectrum-related power spectrum $\mathcal{K}_l^{(2,2)}$ is plotted as function of angular scale l. The left panel is for a noise free ideal all-sky experiment. The right panel correspond to WMAP-V band noise. Various curves correspond to a specific parameter combination of (f_{NL}, g_{NL}) as indicated (see text for details).

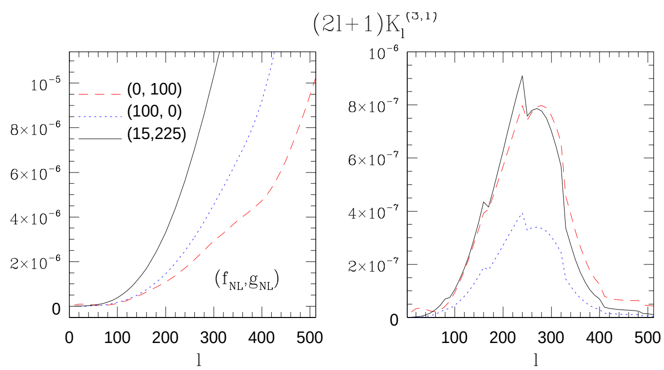

Figure 2. Same as 1 but for the trispectrum-related power spectrum $\mathcal{K}_l^{(3,1)}$ plotted as function of angular scale l for ideal (left panel) and WMAP V-band noise charecteristics.

$$\mathcal{K}_l^{(2,2)} = \sum_{l_i} \frac{T^{l_3 l_4}_{l_1 l_2}(l)\hat{T}^{l_1 l_2}_{l_3 l_4}(l)}{C_{l_1}C_{l_2}C_{l_3}C_{l_4}}; \qquad \mathcal{K}_l^{(3,1)} = \sum_{l_i L} \frac{T^{l_1 l_2}_{l_3 l}(L)\hat{T}^{l_1 l_2}_{l_3 l}(L)}{C_{l_1}C_{l_2}C_{l_3}C_{l}}. \tag{65}$$

Each contribution from a specific configuration formed by various values of l_i and L is weighted by the corresponding C_l to make the estimator optimal. For computation of the variances the fields are however considered as Gaussian, which should be a good approximation as the fields are expected to be close to gaussian. Note that the $C_l s$ here take contributions from both the signal and the noise terms i.e. $C_l = C_l^S + C_l^N$, where C_l^S is just the theoretical expectation for primordial perturbations.
Both $\mathcal{K}_l^{(2,2)}$ and $\mathcal{K}_l^{(3,1)}$ involve both parameters f_{nl} and g_{nl} and can be used for a joint analysis, along with $\mathcal{S}_l^{(2,1)}$ to put constraints from realistic data. It is easy to check that the following relationship holds:

$$\mathcal{K}^{(4)} = \sum_l (2l+1)K_l^{(2,2)} = \sum_l (2l+1)K_l^{(3,1)}. \tag{66}$$

These results therefore generalises the results obtained in (Munshi & Heavens 2009) in the context of bispectral analysis.

5.5 Subtracting the Gaussian or disconnected Contributions

In estimating the trispectrum we need to subtract out the disconnected or Gaussian parts (Hu 2000; Hu & Okamoto 2002). To do this we follow the same procedure but replacing the simulated non-Gaussian maps with their Gaussian counterparts. Both maps should be constructed to have the same power spectrum, and identical noise. If the noise is Gaussian it will only contribute only to the disconnected part. For construction of an estimator which is noise-subtracted we need to follow the same procedure above by constructing Gaussian maps $A^G(r, \hat\Omega), B^G(\hat\Omega, r)$ etc.

$$A^G(\hat\Omega, r) = \sum_{lm} A^G_{lm} Y_{lm}(\hat\Omega); \qquad B^G(\hat\Omega, r) = \sum_{lm} B^G_{lm} Y_{lm}(\hat\Omega); \qquad M^G(\hat\Omega, r) = \sum_{lm} M^G_{lm} Y_{lm}(\hat\Omega);$$

$$A^G_{lm} = \frac{\alpha_{lm}}{C_l} a^G_{lm}; \;\; B^G_{lm} = \frac{\beta_{lm}}{C_l} a^G_{lm}; \;\; M^G_{lm} = \frac{\mu_{lm}}{C_l} a^G_{lm}. \tag{67}$$

We illustrate this using $\mathcal{K}_l^{(2,2)}$; the analysis is very similar for the other estimator. We start by replacing the quantities $\mathcal{I}_l^{AB,AB}(r_1,r_2)$ and $\mathcal{L}_l^{AB,AB}(r_1,r_2)$ by their Gaussian counterparts, $\mathcal{P}_l^{A^GB^G,A^GB^G}(r_1,r_2)$ and $\mathcal{R}_l^{A^GB^G,B^GM^G}(r)$:

$$\mathcal{P}_l^{A^GB^G,A^GB^G}(r_1,r_2) = \frac{1}{2l+1}\sum_m\sum_L F_L(r_1,r_2)A^G(r_1,\hat\Omega)B^G(r_1,\hat\Omega)|_{lm}A^G(r_2,\hat\Omega)B^G(r_2,\hat\Omega)|_{lm}^*$$
$$= \frac{1}{2l+1}\sum_{LM}(-1)^M\sum_m \left\{F_L(r_1,r_2)\alpha_{l_1}(r_1)\beta_{l_2}(r_1)\alpha_{l_2}(r_2)\beta_l(r_2)\right\}\langle a^G_{l_1m_1}a^G_{l_2m_2}a^G_{l_3m_3}a^G_{lm}\rangle\mathcal{G}^{m_1m_2M}_{l_1l_2L}\mathcal{G}^{m_3m_4M}_{l_3l_4L}. \tag{68}$$

These quantities are then used for the construction of an estimator, which in practice aims to estimate the Gaussian contributions to the total trispectra.

$$\mathcal{R}_l^{A^GB^G,B^GM^G}(r) = \frac{1}{2l+1}\sum_m A^G(r,\hat\Omega)B^G(r,\hat\Omega)|_{lm}B^G(r,\hat\Omega)M^G(r,\hat\Omega)|_{lm}^*$$
$$= \frac{1}{2l+1}\sum_m(-1)^m\sum_{l_im_i}\left\{\alpha_{l_1}(r)\alpha_{l_2}(r)\beta_{l_3}(r)\mu_{l_4}(r)\right\}\langle a^G_{l_1m_1}a^G_{l_2m_2}a^G_{l_3m_3}a^G_{l_4m_4}\rangle\mathcal{G}^{m_1m_2m}_{l_1l_2l}\mathcal{G}^{m_3m_4m}_{l_3l_4l}. \tag{69}$$

As before we combine these quantities to form the weighted Gaussian part of the contribution to the trispectrum:

$$\mathcal{G}_l^{(2,2)} = 4f_{nl}^2\int r_1^2dr_1\int r_2^2dr_2\mathcal{P}(r_1,r_2) + 2g_{NL}\int r^2dr\mathcal{R}(r) = \frac{1}{2l+1}\sum_{l_i}\frac{1}{C_{l_1}C_{l_2}C_{l_3}C_{l_4}}\hat{G}^{l_1l_2}_{l_3l_4}(l). \tag{70}$$

Subtracting the Gaussian contribution from the total estimator we can write:

$$\mathcal{K}_l^{(2,2)} = \mathcal{K}_l^{(2,2)} - \mathcal{G}_l^{(2,2)} = \frac{1}{2l+1}\sum_{l_i}\frac{1}{C_{l_1}C_{l_2}C_{l_3}C_{l_4}}T^{l_1l_2}_{l_3l_4}(l)\left\{\hat{T}^{l_1l_2}_{l_3l_4}(l) - \hat{G}^{l_1l_2}_{l_3l_4}(l)\right\}. \tag{71}$$

A very similar calculation for the other fourth-order estimator $\mathcal{G}_l^{3,1}$ provides an identical result. After subtraction of the Gaussian contribution we can write:

$$\mathcal{K}_l^{(3,1)} = \mathcal{K}_l^{(3,1)} - \mathcal{G}_l^{(3,1)} = \frac{1}{2l+1}\sum_{l_i}\sum_L\frac{1}{C_{l_1}C_{l_2}C_{l_3}C_l}T^{l_1l_2}_{l_3l_4}(l)\left\{\hat{T}^{l_1l_2}_{l_3l}(L) - \hat{G}^{l_1l_2}_{l_3l}(L)\right\}. \tag{72}$$

The corresponding one-point collapsed estimator has the following form, and the relationship between the various estimators remains unchanged.

$$\mathcal{K}_l^{(4)} = \sum_{l_i}\sum_L\frac{1}{C_{l_1}C_{l_2}C_{l_3}C_{l_4}}T^{l_1l_2}_{l_3l_4}(l)\left\{\hat{T}^{l_1l_2}_{l_3l_4}(L) - \hat{G}^{l_1l_2}_{l_3l_4}(L)\right\}. \tag{73}$$

In the next section we consider partial sky coverage and the resulting corrections to the Gaussian and non-Gaussian estimators.

6 PARTIAL SKY COVERAGE AND OPTIMISED ESTIMATORS

The terms that are needed for correcting the effect of finite sky coverage and inhomogeneous noise are listed below. These corrections are incorporated by using Monte-carlo simulations of noise realisations (Creminelli 2003; Babich & Zaldarriaga 2004; Babich, Creminelli & Zaldarriaga 2004; Babich 2005; Creminelli et al. 2006, 2007; Creminelli, Senatore, & Zaldarriaga 2007). Whereas in case of bispectral analysis it is just the linear terms which are needed for corrections, for trispectral analysis there are both linear and quadratic terms.

We will treat the general case in later subsections, but first we present results for the simpler case of homogeneous noise and for high wavenumbers where the density of uncorrelated states is modified by inclusion of the fraction of sky observed, f_{sky}.

6.1 Corrective terms for the Estimator $\mathcal{K}_l^{3,1}$ in the absence of spherical symmetry

$$\hat{\mathcal{J}}_l^{A_1B_1B_2,A_2} = \frac{1}{f_{sky}}\left[\tilde{\mathcal{J}}_l^{A_1B_1B_2,A_2} - \mathcal{I}_l^{\rm Lin} - \mathcal{I}_l^{\rm Quad}\right] \tag{74}$$

$$\mathcal{I}_l^{\rm Lin} = \frac{1}{f_{sky}}\left[\mathcal{J}_l^{\langle A_1B_1\rangle B_2,A_2} + \mathcal{J}_l^{\langle A_1B_2\rangle B_1,A_2} + \mathcal{J}_l^{\langle B_1B_2\rangle A_1,A_2} + \mathcal{J}_l^{A_1B_1\langle B_2,A_2\rangle} + \mathcal{J}_l^{A_1B_2\langle B_1,A_2\rangle} + \mathcal{J}_l^{B_1B_2\langle A_1,A_2\rangle}\right] \tag{75}$$

$$\mathcal{I}_l^{\rm Quad} = \frac{1}{f_{sky}}\left[\mathcal{J}_l^{\langle A_1B_1B_2\rangle,A_2} + \mathcal{J}_l^{A_1\langle B_1B_2,A_2\rangle} + \mathcal{J}_l^{B_1\langle A_1B_2,A_2\rangle} + \mathcal{J}_l^{B_2\langle A_1B_1,A_2\rangle}\right]. \tag{76}$$

The expressions are similar for the other terms that depend on only one radial distance:

$$\hat{\mathcal{L}}_l^{ABB,M} = \frac{1}{f_{sky}}\left[\tilde{\mathcal{L}}_l^{ABB,M} - \mathcal{I}_l^{Lin} - \mathcal{I}_l^{\rm Quad}\right] \tag{77}$$

$$\mathcal{I}_l^{\rm Lin} = \frac{1}{f_{sky}}\left[\mathcal{L}_l^{A\langle BB,M\rangle} + 2\mathcal{L}_l^{B\langle A,M\rangle} + 2\mathcal{L}_l^{B\langle AB\rangle,M}\right] \tag{78}$$

$$\mathcal{I}_l^{\rm Quad} = \frac{1}{f_{sky}}\left[\mathcal{L}_l^{A\langle BB\rangle,M} + 2\mathcal{L}_l^{B\langle AB\rangle,M}\right]. \tag{79}$$

To simplify the presentation we have used the symbol $A(r_1,\hat\Omega) = A_1; A(r,\hat\Omega) = A$ and so on. Essentially we can see that there are terms which are linear in the input harmonics and terms which are quadratic in the input harmonics. The terms which are linear are also proportional to the bispectrum of the remaining 3D fields which are being averaged. On the other hand the prefactors for quadratic terms are 3D correlation functions of the remaining two fields. Finally putting all of these expressions we can write:

$$\tilde{\mathcal{K}}_l^{(3,1)} = 4f_{\rm NL}\int r_1^2 dr_1 \int r_2 dr_2 \tilde{\mathcal{J}}_l^{AB^2,A}(r_1,r_2) + \int r^2 dr \tilde{\mathcal{L}}_l^{AB^2M}(r). \tag{80}$$

From a computational point of view clearly the overlap integral $F_L(r_1,r_2)$ will be expensive and may determine to what resolution ultimately these direct techniques can be implemented. Use of these techniques directly involving Monte-Carlo numerical simulations will be dealt with in a separate paper (Smidt et al. in preparation). To what extent the linear and quadratic terms are important in each of these contributions can only be decided by testing against simulation.

6.2 Corrective terms for the Estimator $\mathcal{K}_l^{2,2}$ in the absence of Spherical symmetry

The unbiased estimator for the other estimator can be constructed in a similar manner. As before there are terms which are quadratic in input harmonics with a prefactor proportional to terms involving cross-correlation or variance of various combinations of 3D fields and there will be linear terms (linear in input harmonics) with a prefactor proportional to bispectrum associated with various 3D fields.

$$\hat{\mathcal{J}}_l^{A_1B_1,A_2B_2} = \frac{1}{f_{sky}}\left[\tilde{\mathcal{J}}_l^{A_1B_1,A_2B_2} - I_l^{\rm Lin} - I_l^{\rm Quad}\right] \tag{81}$$

$$\mathcal{I}^{\rm Lin} = \frac{1}{f_{sky}}\left[\mathcal{J}_l^{A_1B_1,\langle A_2B_2\rangle} + \mathcal{J}_l^{\langle A_1B_1\rangle,A_2B_2} + \mathcal{J}_l^{A_1\langle B_1,B_2\rangle A_2} + \mathcal{J}_l^{B_2\langle A_2,B_2\rangle A_2} + \mathcal{J}_l^{B_2\langle A_2,A_2\rangle B_2} + \mathcal{J}_l^{A_2\langle B_2,A_2\rangle B_2}\right] \tag{82}$$

$$\mathcal{I}^{\rm Quad} = \frac{1}{f_{sky}}\left[\mathcal{J}_l^{A_1\langle B_1,A_2B_2\rangle} + \mathcal{J}_l^{B_1\langle A_1,A_2B_2\rangle} + \mathcal{J}_l^{\langle A_1B_1,A_2\rangle B_2} + \mathcal{J}_l^{\langle A_2B_2,B_2\rangle A_2}\right]. \tag{83}$$

The terms such as $\mathcal{K}_l^{AB,BM}(r_1,r_2)$ can be constructed in a very similar way. We display the term $\mathcal{K}_l^{AB^2,A}(r_1,r_2)$ with all its corrections included.

$$\hat{\mathcal{L}}_l^{AB,BM} = \frac{1}{f_{sky}}\left[\tilde{\mathcal{L}}_l^{AB,BM} - I_l^{\rm Lin} - I_l^{\rm Quad}\right] \tag{84}$$

$$I_l^{\rm Quad} = \frac{1}{f_{sky}}\left[\mathcal{L}_l^{A\langle B^2\rangle,A} + 2\mathcal{L}_l^{AB\langle B,A\rangle} + \mathcal{K}_l^{B^2\langle A,A\rangle} + 2\mathcal{K}_l^{B\langle AB,A\rangle} + \mathcal{K}_l^{A\langle B^2,A\rangle}\}\right] \tag{85}$$

$$I_l^{\rm Lin} = \frac{1}{f_{sky}}\left[\mathcal{J}_l^{A_1\langle B_1,B_2M_2\rangle} + \mathcal{J}_l^{B_1\langle A_1,B_2M_2\rangle} + \mathcal{J}_l^{\langle A_1B_1,M_2\rangle B_2} + \mathcal{J}_l^{\langle A_2B_2,B_2\rangle M_2}\right]. \tag{86}$$

The importance of the linear terms greatly depends on the target model being considered. For example, while linear terms for bispectral analysis can greatly reduce the amount of scatter in the estimator for *local* non-Gaussianity, the linear term is less important in modelling the *equilateral* model. In any case the use of such Monte-Carlo (MC) maps is known to reduce the scatter and can greatly simplify the estimation of non-Gaussianity. This can be useful, as fully optimal analysis with inverse variance weighting, which treats mode-mode coupling completely, may only be possible on low-resolution maps.

6.3 Corrective terms for the Estimator $\mathcal{K}^{(4)}$ in the absence of spherical symmetry

The corrections to the optimised one-point fourth order cumulant can also be analysed in a very similar manner. It is expected that higher-order cumulants will be more affected by partial sky coverage and loss of spherical symmetry because of inhomogeneous noise.

$$\hat{\mathcal{K}}^{(4)} = \frac{1}{f_{sky}}[\tilde{\mathcal{K}}^{(4)} - I^{\rm Lin} - I^{\rm Quad}\Big]. \tag{87}$$

The linear and quadratic terms can be expressed in terms of MC averages. We list terms which involve the overlap integral and the one with single line-of-sight integration.

$$I^{\rm Lin} = 4f_{\rm NL}\int r_1^2 dr_1 \int r_2 dr_2 \sum_L F_L^{12}\left\{A_1\langle B_1A_2B_2\rangle + B_1\langle A_1A_2B_2\rangle + A_2\langle A_1B_1A_2\rangle + B_2\langle A_1B_1A_2\rangle\right\}$$

Table 1. Various multispectra and associated power-spectra are tabulated along with their cosmological use

Order	Cumulants & Correlator	Power Spectra	Use
$2 = (1+1)$	$\langle \delta^2 T(\hat\Omega)\rangle$, $\langle \delta T(\hat\Omega)\delta T(\hat\Omega')\rangle$	C_l	Constraints on Cosmology $(\Omega, H_0 \dots)$
$3 = (2+1)$	$\langle \delta^3 T(\hat\Omega)\rangle$, $\langle \delta^2 T(\hat\Omega)\delta T(\hat\Omega')\rangle$	$S^{(3)},\ S_l^{(2,1)}$	Inflationary Models $f_{\rm NL}$, Secondaries x Lensing b_l
$4 = (3+1), (2+2)$	$\langle \delta^4 T(\hat\Omega)\rangle$, $\langle \delta^3 T(\hat\Omega)\delta(\hat\Omega')\rangle$,$\langle \delta^2 T(\hat\Omega)\delta^2 T(\hat\Omega')\rangle$	$K^{(4)},\ K_l^{(2,2)}$, $K_l^{(3,1)}$	$f_{\rm NL}, g_{\rm NL}$, KSZ-Lensing Sep. Internal Lensing det.

$$+2g_{\rm NL}\int r^2 dr \left\{ A\langle B^2 M\rangle + 2B\langle ABM\rangle + M\langle AB^2\rangle \right\} \tag{88}$$

$$I^{\rm Quad} = 4f_{\rm NL}\int r_1^2 dr_1 \int r_2 dr_2 \sum_L F_L^{12}\left\{A_1B_1\langle A_2B_2\rangle + A_2B_2\langle A_1B_1\rangle + A_1B_2\langle A_2B_1\rangle + A_2B_1\langle A_1B_2\rangle\}\right\rangle$$

$$+2g_{\rm NL}\int r^2 dr \left\{ AM\langle B^2\rangle + B^2\langle AM\rangle + 2BM\langle AB\rangle \right\}. \tag{89}$$

We have introduced the shorthand notation $F_L^{12} = F_L(r_1, r_2)$.

7 REALISTIC SURVEY STRATEGIES: EXACT ANALYSIS

In this section, we include the full optimisation, including the mode-mode coupling introduced by partial sky coverage and inhomogeneous noise, generalising results from the three-point level (Babich 2005; Smith & Zaldarriaga 2006; Smith, Zahn & Dore 2007; Smith, Senatore & Zaldarriaga 2009). As before, we find that for the trispectrum, the addition of quadratic terms in addition to linera terms is needed. The analysis presented here is completely generic and will not depend on details of factorizability properties of the trispectra. For any specific form of the trispectrum the technique presented here can always provide optimal estimators. Importantly, this makes it suitable also for the study of the trispectrum contribution from secondaries, and offers the possibility of determining whether any observed connected trispectrum is primordial or not. Generalisation to multiple sources of trispectrum is straightforward, following Smidt et al. (2009)

Trispectra from secondary anisotropies such as gravitational lensing are expected to dominate the contribution from primary anisotropies (Cooray & Kesden 2003). The estimator we develop here will be directly applicable to data from various surveys, but the required direct inversion of the covariance matrix in the harmonic domain may not be computationally feasible in near future. Nevertheless an exact analysis may still be beneficial for low-resolution degraded maps where the primary anisotropy dominates. At higher resolution the exact analysis will reduce to the one discussed in previous sections. In addition it may be possible at least to certain resolution to bypass the exact inverse variance weighting by introducing a conjugate gradient techniques.

The general theory for optimal estimation from data was developed by Babich (2005) for the analysis of the bispectrum, and was later implemented by Smith & Zaldarriaga (2006). For arbitrary sky coverage and inhomogeneous noise the estimator will be fourth order in input harmonics, and involves matched filtering to maximise the response of the estimator when the estimated trispectrum matches theoretical expectation. We present results for both one-point cumulants and two-point cumulant correlators or power spectra associated with trispectra. The estimators presented here, $E_l^{(3,1)}$ and $E_l^{(2,2)}$ are generalisations of estimators $C_l^{(3,1)}$, $C_l^{(2,2)}$ and $\mathcal{K}^{(3,1)}$ and $\mathcal{K}^{(2,2)}$ presented in previous sections.

7.1 One-point Estimators

We will use inverse variance weighting harmonics recovered from the sky. The covariance matrix, expressed in the harmonics domain, $C^{-1}_{LM,lm}$ when used to filter out modes recovered directly from sky are a_{lm} are denoted as A_{LM}. We use these harmonics to construct optimal estimators. For all-sky coverage and homogeneous noise, we can recover $A_{lm} = a_{lm}/C_l$, with C_ls including signal and noise. We start by keeping in mind that the trispectrum can be expressed in terms of the harmonic transforms a_{lm} as follows:

$$T^{l_al_b}_{l_cl_d}(L) = (2l+1)\sum_{m_i}\sum_M (-1)^M \begin{pmatrix} l_a & l_b & L \\ m_a & m_b & M \end{pmatrix}\begin{pmatrix} l_c & l_d & L \\ m_c & m_d & -M \end{pmatrix} a_{l_am_a}\dots a_{l_dm_d} \qquad (i\in a,b,c,d). \tag{90}$$

Based on this expression we can devise a one-point estimator. In our following discussion, the relevant harmonics can be based on partial sky coverage.

$$Q^{(4)}[a] = \frac{1}{4!}\sum_{LM}(-1)^M\sum_{l_im_i}\Delta(l_i;L)\,T^{l_1l_2}_{l_3l_4}(L)\begin{pmatrix} l_a & l_b & L \\ m_a & m_b & M \end{pmatrix}\begin{pmatrix} l_c & l_d & L \\ m_c & m_d & -M \end{pmatrix} a_{l_1m_1}\dots a_{l_4m_4} \tag{91}$$

The term $\Delta(l_i, L)$ is introduced here to avoid contributions from Gaussian or disconnected contributions. $\Delta(l_i, L)$ vanishes if amy pair of l_is becomes equal or $L = 0$ which effectively reduces the trispectra to a product of two power spectra (i.e. disconnected Gaussian pieces).

We will also need the first-order and second-order derivative with respect to the input harmonics. The linear terms are proportional to the first derivatives and the quadratic terms are proportional to second derivatives of the function $Q[a]$, which is quartic in input harmonics.

$$\partial_{lm} Q^{(4)}[a] = \frac{1}{3!}\sum_{LM}(-1)^M \sum_{l_i m_i} \Delta(l_i;L)\, T^{l_1 l_2}_{l_3 l_4}(L) \begin{pmatrix} l & l_a & L \\ m & m_a & M \end{pmatrix} \begin{pmatrix} l_b & l_c & L \\ m_b & m_c & -M \end{pmatrix} a_{l_a m_b} a_{l_a m_b} a_{l_c m_c}. \tag{92}$$

The first-order derivative term $\partial_{lm} Q^{(4)}[a]$ is cubic in input maps and the second order derivative is quadratic in input maps (harmonics). However unlike the estimator itself $Q^{(4)}[a]$, which is simply a number, these objects represent maps constructed from harmonics of the observed maps.

$$\partial_{lm}\partial_{l'm'} Q^{(4)}[a] = \frac{1}{2!}\sum_{LM}(-1)^M \sum_{l_i m_i} \Delta(l_i;L)\, T^{l_1 l_2}_{l_3 l_4}(L) \begin{pmatrix} l & l_a & L \\ m & m_a & M \end{pmatrix} \begin{pmatrix} l & l_b & L \\ m & m_b & -M \end{pmatrix} a_{l_a m_a} a_{l_b m_b}. \tag{93}$$

The optimal estimator for the one-point cumulant can now be written as follows. This is optimal in the presence of partial sky coverage and most general inhomogeneous noise:

$$E^{(4)}[a] = \frac{1}{N}\left\{ Q^{(4)}[C^{-1}a] - [C^{-1}a]_{lm}\langle \partial_{lm} Q^{(4)}[A]\rangle - [C^{-1}a]_{lm}[C^{-1}a]_{l'm'}\langle \partial_{lm}\partial_{l'm'} Q^{(4)}[C^{-1}a]\rangle \right\}. \tag{94}$$

The terms which are subtracted out are linear and quadratic in input harmonics. The linear term is similar to the one which is used for bispectrum estimation, whereas the quadratic terms correspond to the disconnected contributions and will vanish identically as we have designed our estimators in such a way that it will not take any contribution from disconnected Gaussian terms. The Fisher matrix reduces to a number which we have used for normalisation.

$$F = \frac{1}{N} = \quad \frac{1}{4!}\sum_{LM}\sum_{L'M'}\sum_{(all\ lm)}\sum_{(all\ l'm')} (-1)^M(-1)^{M'}\, \Delta(l_i;L)\Delta(l'_i;L')\, T^{l_a l_b}_{l_c l_c}(L) T^{l_{a'} l_{b'}}_{l_{c'} l_{d'}}(L) \begin{pmatrix} l_a & l_b & L \\ m_a & m_b & M \end{pmatrix} \begin{pmatrix} l_c & l_d & L \\ m_c & m_d & -M \end{pmatrix}$$
$$\times \begin{pmatrix} l'_a & l'_b & L' \\ m'_a & m'_b & M' \end{pmatrix} \begin{pmatrix} l'_c & l'_d & L' \\ m'_c & m'_d & -M' \end{pmatrix} C^{-1}_{l_a m_a, l'_a m'_a} \dots C^{-1}_{l_d m_d, l'_d m'_d}. \tag{95}$$

The ensemble average of this one-point estimator will be a linear combination of parameters $f^2_{\rm NL}$ and $g_{\rm NL}$. Estimators constructed at the level of three-point cumulants (Smith & Zaldarriaga 2006; Munshi & Heavens 2009) can be used jointly with this estimator to put independent constraints separately on $f_{\rm NL}$ and $g_{\rm NL}$. As discussed before, while one-point estimator has the advantage of higher signal-to-noise, such estimators are not immune to contributions from an unknown component which may not have cosmological origin, such as inadequate foreground separation. The study of these power spectra associated with bispectra or trispectra can be useful in this direction. Note that these direct estimators are computationally expensive due to the inversion and multiplication of large matrices, but can be implemented in low-resolution studies where primordial signals may be less contaminated by foreground contributions or secondaries.

7.2 Two-point Estimators: Power Spectra associated with Trispectra

Generalising the above expressions for the case of the power spectrum associated with trispectra, we recover the two power spectra we have discussed in previous sections. The information content in these power spectra are optimal, and when summed for over L we can recover the results of one-point estimators.

$$Q^{(2,2)}_L[a] = \frac{1}{4!}\sum_{M}(-1)^M \sum_{l_i m_i} T^{l_a l_b}_{l_c l_d}(L) \begin{pmatrix} l_a & l_b & L \\ m_a & m_b & M \end{pmatrix} \begin{pmatrix} l_c & l_d & L \\ m_c & m_d & -M \end{pmatrix} a_{l_a m_a}\dots a_{l_d m_d}. \tag{96}$$

The derivatives at first order and second order are as series of maps (for each L) constructed from the harmonics of the observed sky. These are used in the construction of linear and quadratic terms.

$$\partial_{lm} Q^{(2,2)}_L[a] = \frac{1}{3!}\sum_{T}(-1)^M \sum_{l_i m_i} \Delta(l_i;L) T^{l_a l_b}_{l_c l_d}(L) \begin{pmatrix} l & l_a & L \\ m & m_b & M \end{pmatrix} \begin{pmatrix} l_b & l_c & L \\ m_b & m_c & -M \end{pmatrix} a_{l_a m_a}\dots a_{l_c m_c}. \tag{97}$$

We can construct the other estimator in a similar manner. To start with we define the function $Q^{(3,1)}_L[a]$ and construct its first and second derivatives. These are eventually used for construction of the estimator $E^{(3,1)}_L[a]$. As we have seen, both of these estimators can be collapsed to a one-point estimator $Q^{(4)}[a]$. As before, the variable a here denotes input harmonics a_{lm} recovered from the noisy observed sky.

$$Q^{(3,1)}_L[a] = \frac{1}{3!}\sum_{M}\sum_{ST}(-1)^T a_{LM} \sum_{l_i m_i} \Delta(l_i,L;T) T^{L l_b}_{l_c l_d}(T) \begin{pmatrix} L & l_b & S \\ M & m_b & T \end{pmatrix} \begin{pmatrix} l_c & l_d & S \\ m_c & m_d & -T \end{pmatrix} a_{l_b m_b}\dots a_{l_d m_d}. \tag{98}$$

The derivative term will have two contributing terms corresponding to the derivative w.r.t. the free index $\{LM\}$ and the terms where indices are summed over e.g. $\{lm\}$, which is very similar to the results for the bispectrum analysis with the estimator $Q^{(2,1)}_L[a]$. One major difference that needs to be taken into account is the subtraction of the Gaussian contribution. The function $\Delta(l_i,L)$ takes into account of this subtraction.

$$\partial_{lm} Q^{(3,1)}_L[a] = \quad \sum_{ST}\sum_{l_i m_i} \Delta(l_i,L;T) T^{L l_2}_{l_3 l_4}(T) \begin{pmatrix} l & l_b & S \\ m & m_b & T \end{pmatrix} \begin{pmatrix} l_c & l_d & S \\ m_c & m_d & -T \end{pmatrix} a_{l_b m_b}\dots a_{l_d m_d} \tag{99}$$

$$+3\sum_M\sum_{l_im_i} a_{LM}\sum_T \begin{pmatrix} L & l & S \\ M & m & T \end{pmatrix}\begin{pmatrix} l_b & l_c & S \\ m_b & m_c & -T \end{pmatrix} a_{l_bm_b}a_{l_cm_c}. \tag{100}$$

Using these derivatives we can construct the estimators $E_L^{(3,1)}$ and $E_L^{(2,2)}$:

$$E_L^{(3,1)} = N_{LL'}^{-1}\left\{Q_{L'}^{(3,1)}[C^{-1}a] - [C^{-1}a]_{lm}\langle\partial_{lm}Q_{L'}^{(3,1)}[C^{-1}a]\rangle\right\} \tag{101}$$

$$E_L^{(2,2)} = N_{LL'}^{-1}\left\{Q_{L'}^{(2,2)}[C^{-1}a] - [C^{-1}a]_{lm}\langle\partial_{lm}Q_{L'}^{(2,2)}[C^{-1}a]\rangle\right\}, \tag{102}$$

where summation over L' is implied. The quadratic terms will vanish, as they contribute only to the disconnected part. The normalisation constants are the Fisher matrix elements $F_{LL'}$ which can be expressed in terms of the target trispectra $T^{l_1l_2}_{l_3l_4}(L)$ and inverse covariance matrices C^{-1} used for the construction of these estimators. The Fisher matrix for the estimator $E_L^{(3,1)}$, i.e. $F_{LL'}^{(3,1)}$ can be expressed as:

$$\begin{aligned}
[N^{-1}]_{LL'} = F_{LL'}^{(2,2)} &= \left(\frac{1}{4!}\right)^2 \sum_{ST,S'T'}\sum_{(all\ lm,l'm')} (-1)^M(-1)^{M'}[T^{l_al_b}_{l_cl_c}(L)][T^{l_{a'}l_{b'}}_{l_{c'}l_{d'}}(L')] \\
&\Delta(l_il;L)\Delta(l_i';L') \times \begin{pmatrix} l_a & l_b & S \\ m_a & m_b & T \end{pmatrix}\begin{pmatrix} L & l_d & S \\ M & m_d & -T \end{pmatrix}\begin{pmatrix} l_a' & l_b' & S' \\ m_a' & m_b' & T' \end{pmatrix}\begin{pmatrix} L' & l_d' & S' \\ M' & m_d' & -T' \end{pmatrix} \\
&\times\Big\{6C^{-1}_{LM,L'M'}C^{-1}_{l_am_a,l_a'm_a'}C^{-1}_{l_bm_b,l_b'm_b'}C^{-1}_{l_cm_c,l_c'm_c'} + 18C^{-1}_{LM,l_a'm_a'}C^{-1}_{l_am_a,L'M'}C^{-1}_{l_bm_b,l_b'm_b'}C^{-1}_{l_cm_c,l_c'm_c'}\Big\}.
\end{aligned} \tag{103}$$

Similarly for the other estimator $E_L^{(2,2)}$, the Fisher matrix $F_{LL'}^{(2,2)}$ can be written as a function of the associated trispectrum and the covariance matrix of various modes. For further simplification of these expressions we need to make simplifying assumptions for a specific type of trispectra, see Munshi & Heavens (2009) for more details for such simplifications in the bispectrum.

$$\begin{aligned}
[N^{-1}]_{LL'} = F_{LL'}^{(3,1)} = \left(\frac{1}{4!}\right)^2 \sum_M\sum_{M'}\sum_{(all\ lm)}\sum_{(all\ l'm')} (-1)^M(-1)^{M'}T^{l_al_b}_{l_cl_c}(L)T^{l_{a'}l_{b'}}_{l_{c'}l_{d'}}(L')\begin{pmatrix} l_a & l_b & L \\ m_a & m_b & M \end{pmatrix}\begin{pmatrix} L & l_d & L \\ M & m_d & -M \end{pmatrix} \\
\times\begin{pmatrix} l_a' & l_b' & L' \\ m_a' & m_b' & M' \end{pmatrix}\begin{pmatrix} L' & l_d' & L' \\ M' & m_d' & -M' \end{pmatrix}\Delta(l_il;L)\Delta(l_i';L')\, C^{-1}_{l_am_a,l_a'm_a'}\dots C^{-1}_{l_dm_d,l_d'm_d'}.
\end{aligned} \tag{104}$$

Knowledge of sky coverage and the noise characteristics resulting from a specific scanning strategy etc. is needed for modelling of $C^{-1}_{l_dm_d,l_d'm_d'}$. We will discuss the impact of inaccurate modelling of the covariance matrix in the next section. The direct summation we have used for the construction of the Fisher matrix may not be feasible except for low resolution studies. However a hybrid method may be employed to combine the estimates from low resolution maps using exact method with estimates from higher resolution using other faster but optimal techniques described in previous section. In certain situations when the data is noise-dominated further approximation can be made to simplify the implementation. A more detailed discussion will be presented elsewhere.

7.3 Approximation to exact C^{-1} weighting and non-optimal weighting

If the covariance matrix is not accurately known due to the lack of exact beam or noise characteristics, or due to limitations on computer resources, it can be approximated. An approximation R of C^{-1} then acts as a regularisation method. The corresponding generic estimator can then be expressed as:

$$\hat{E}_L^Z[a] = \sum_{L'}[F^{-1}]_{LL'}\left\{Q_{L'}^Z[Ra] - [Ra]_{lm}\langle\partial_{lm}Q_{L'}^Z[Ra]\rangle\right\}; \qquad Z\in\{(2,2),(3,1)\}. \tag{105}$$

As before we have assumed sums over repeated indices and $\langle\cdot\rangle$ denotes Monte-Carlo (MC) averages. As evident from the notations, the estimator above can be of type $E_L^{(3,1)}$ or $E_L^{(2,2)}$. For the collapsed case $E_L^{(4)}$ can also be handled in a very similar manner.

$$\hat{E}[a] = [F^{-1}]_{LL'}\left\{Q_L[Ra] - [Ra]_{lm}\langle\partial_{lm}Q_L[Ra]\rangle\right\}. \tag{106}$$

We will drop the superscript Z for simplicity but any conclusion drawn below will be valid for both specific cases i.e. $Z\in\{(2,2),(3,1)\}$. The normalisation constant which acts also as inverse of associated Fisher matrix $F_{LL'}$ can be written as:

$$F_{LL'} = \langle(\hat{E}_L)(\hat{E}_{L'})\rangle - \langle(\hat{E}_L)\rangle\langle(\hat{E}_{L'})\rangle = \frac{1}{4}\langle\partial_{lm}Q_L[Ra]\partial_{lm}Q_L[Ra]\rangle - \frac{1}{4}\langle\partial_{lm}Q_L[Ra]\rangle\langle\partial_{lm}Q_{L'}[Ra]\rangle. \tag{107}$$

The construction of $F_{LL'}$ is equivalent to the calculation presented for the case of $R = C^{-1}$. For one-point estimator we similarly can write $F^R = \sum_{LL'}F^R_{LL'}$. The optimal weighting can be replaced by arbitrary weighting. As a spacial case we can also use no weighting at all $R = I$. Although the estimator remains unbiased the scatter however increases as the estimator is no longer optimal. Use of arbitrary weights makes the estimator equivalent to a PCL estimator.

7.4 Joint Estimation of Multiple trispectra

It may be of interest to estimate several trispectra jointly. In such scenarios it is indeed important to construct a joint Fisher matrix which will

$$\hat{E}^X_L[a] = \sum_{XY}\sum_{LL'}[F^{-1}]^{XY}_{LL'}\hat{E}^Y_{L'}[a]. \tag{108}$$

The estimator $\hat{E}^X_L[a]$ is generic and it could be either $E^{(3,1)}$ or $E^{(2,2)}$. Here X and Y corresponds to different trispectra of type X and Y, these could be e.g. primordial trispectra from various inflationary scenarios. It is possible of course to do a joint estimation of primary and secondary trispectra. The off-diagonal blocks of the Fisher matrix will correspond to cross-talks between various types of bispectra. Indeed principal component analysis or Generalised eigenmode analysis can be useful in finding how many independent components of such trispectra can be estimated from the data.

The cross terms in the Fisher matrix elements will be of following type:

$$F^{XY}_{LL'} = \left(\frac{1}{4!}\right)^2 \sum_{ST,S'T'}\sum_{(all\ lm,l'm')}(-1)^M(-1)^{M'}[T^{l_al_b}_{l_cl_c}(L)]^X[T^{l_{a'}l_{b'}}_{l_{c'}l_{d'}}(L')]^Y$$
$$\times\begin{pmatrix} l_a & l_b & S \\ m_a & m_b & T\end{pmatrix}\begin{pmatrix} L & l_d & S \\ M & m_d & -T\end{pmatrix}\begin{pmatrix} l'_a & l'_b & S' \\ m'_a & m'_b & T'\end{pmatrix}\begin{pmatrix} L' & l'_d & S' \\ M' & m'_d & -T'\end{pmatrix}$$
$$\times\Big\{6C^{-1}_{LM,L'M'}C^{-1}_{l_am_a,l'_am'_a}C^{-1}_{l_bm_b,l'_bm'_b}C^{-1}_{l_cm_c,l'_cm'_c} + 18C^{-1}_{LM,l'_am'_a}C^{-1}_{l_am_a,L'M'}C^{-1}_{l_bm_b,l'_bm'_b}C^{-1}_{l_cm_c,l'_cm'_c}\Big\}. \tag{109}$$

The expression displayed above is valid only for $E^{(3,1)}$, exactly similar results holds for the other estimator $E^{(3,1)}$. For $X = Y$ we recover the results presented in previous section for independent estimates. As before we recover the usual result for one-point estimator for Q^4 from the Fisher matrix of $Q^{(3,1)}_L$ or $Q^{(2,2)}_L$, with corresponding estimator modified accordingly.

$$F^{XY} = \sum_{LL'}F^{XY}_{LL'}; \qquad \hat{E}^X[a] = \sum_{XY}[F^{-1}]^{XY}\hat{E}^Y[a]. \tag{110}$$

A joint estimation can provide clues to cross-contamination from different sources of trispectra. It also provide information about the level of degeneracy involved in such estimates.

8 CONCLUSIONS

In the near future the all-sky Planck satellite will complete mapping the CMB sky in unprecedented detail, covering a huge frequency range. The cosmological community will have the opportunity to use the resulting data to constrain available theoretical models. While the power spectrum provides the bulk of the information, going beyond this level will lift degeneracies among various early universe scenarios which otherwise have near identical power spectra. The higher-order spectra are the harmonic transforms of multi-point correlation functions, which contain information which can be difficult to extract using conventional techniques. This is related to their complicated response to inhomogeneous noise and sky coverage. A practical advance is to form collapsed two-point statistics, constructed from higher-order correlations, which can be extracted using conventional power-spectrum estimation methods.

We have specifically studied and developed three different types of estimators which can be employed to analyse these power spectra associated with higher-order statistics. The MASTER-based approach (Hivon et al. 2001) is typically employed to estimate pseudo-C_ls from the masked sky in the presence of noise; also see Efstathiou (2004, 2006). These are unbiased estimators but the associated variances and scatter can be estimated analytically with very few simplifying assumptions. We extend these estimators to study higher-order correlation functions. We develop estimators for $C^{(2,1)}_l$ for the skew-spectrum (3-point) as well as $C^{(3,1)}_l$ and $C^{(2,2)}_l$ which are power spectra of fields related to the trispectrum, or kurt-spectrum (4-point). The removal of the Gaussian contribution is achieved by applying the same estimators to a set of Gaussian simulations with identical power spectra and subtracting.

As a next step we generalised the estimators employed by Yadav, Komatsu,& Wandelt (2007); Yadav & Wandelt (2008); Yadav et al. (2008) and others to study the kurt-spectrum. These methods are computationally expensive and can be implemented using a Monte-Carlo pipeline which can generate 3D maps from the cut-sky harmonics using radial integrations of a target theoretical model. The Monte-Carlo generation of 3D maps is the most computationally expensive part and dominates the calculation. The technique nevertheless has been used extensively, as it remains highly parallelisable and is optimal in the presence of homogeneous noise and near all-sky coverage. The corrective terms involves linear and quadratic terms for lack of spherical symmetry due to inhomogeneous noise and partial sky coverage. These terms can be computed using a Monte-Carlo chain. We also showed that the radial integral involved at the three-point analysis needs to extended to a double-integral for the trispectrum. The speed of this analysis depends on how fast we can generate non-Gaussian maps. It is possible also to use the same formalism to study cross-contamination from other sources such as point sources or other secondaries while also determining primordial non-Gaussianity. The analysis also allows us to compute the overlap or degeneracy among various theoretical models for primordial non-Gaussianity.

Finally we presented the analysis for the case of estimators which are completely optimal even in the presence of inhomogeneous noise and arbitrary sky coverage (Smith & Zaldarriaga 2006). Extending previous work by Munshi & Heavens (2009) which concentrated only on the skew-spectrum, we showed how to generalise to the trispectrum. This involves finding a fast method to construct and invert the covariance matrix $C_{lml'm'}$ in multipole space. In most practical circumstances it is possible only to find an approximation to the exact covariance matrix, and to cover this we present analysis for an approximate matrix which can be used as instead of C^{-1}. This makes the method marginally suboptimal but it remains unbiased. The four-point correlation function also takes contributions which are purely Gaussian in nature. The subtraction of these contributions is again simplified by the use of Gaussian Monte-Carlo maps with the same power spectrum. A Fisher analysis was presented for the construction of the error covariance matrix, allowing

joint estimation of trispectra contributions from various sources, primaries or secondaries. Such a joint estimation give us fundamental limits on how many sources of non-Gaussianity can be jointly estimated from a specific experimental set up. A more detailed Karhuenen-Loéve eigenmode analysis will be presented elsewhere. The detection of non-primordial effects, such as weak lensing of the CMB, the kinetic SZ effect, the Ostriker-Vishniac effect and the thermal SZ effect can provide valuable additional cosmological information (Riquelmel & Spergel 2006). The detection of CMB lensing at the level of the bispectrum needs external data sets which trace large-scale structures, but the trispectrum offers an internal detection, albeit with reduced S/N, providing a valuable consistency check.

At the level of the bispectrum, primordial non-Gaussianity can for many models be described by a single parameter f_{NL}. The two degenerate kurt-spectra (power spectra related to tri-spectrum) we have studied at the 4-point level require typically two parameters such as f_{NL} and g_{NL}. Use of the two power spectra will enable us to put separate constraints on f_{NL} and g_{NL} without using information from lower-order analysis of bispectrum, but they can all be used in combination. Clearly at even higher-order more parameters will be needed to describe various parameters $(f_{NL}, g_{NL}, h_{NL}, \dots)$ which will all be essential in describing degenerate sets of power spectra associated with multispectra at a specific level. Note we should keep in mind that higher-order spectra will be progressively more dominated by noise and may not provide useful information beyond a certain point.

The power of the estimators we have constructed largely depends on finding a techniques to simulate non-Gaussian maps with a specified bispectrum and trispectrum. We have discussed the possibility of using our technique to generate all-sky CMB maps with specified lower-order spectra, i.e. power spectrum, bispectra and trispectra. These will generalise previous results by Smith & Zaldarriaga (2006) which can guarantee a specific form at bispectrum level.

While we have primarily focused on CMB studies, our estimators can also be useful in other areas e.g. studies involving 21cm studies, near all-sky redshift surveys and weak lensing surveys. The estimators described here can be useful for testing theories for primordial and/or gravity-induced secondary trispectrum using such diverse data sets. In the near future all-sky CMB experiments such as Planck can provide maps covering a huge frequency range and near-all sky coverage. The estimators described here can play a valuable role in analysing such maps.

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APPENDIX A: 3J SYMBOLS

$$\sum_{l_3 m_3} (2l_3+1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1' & m_2' & m \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \tag{A1}$$

$$\sum_{m_1 m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{l_3 l_3'} \delta_{m_3 m_3'}}{2l_3+1} \tag{A2}$$

$$(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} = \frac{(-1)^l}{\sqrt{(2l+1)}} \delta_{l' 0} \tag{A3}$$